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HOMOLOGY OF THE CONFIGURATION SPACES OF QUASI-EQUILATERAL POLYGON LINKAGES

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ABSTRACT. We consider the configuration space $M_{n,r}$ of quasi-equilateral polygon linkages with n vertices each edge having length 1 except for one fixed edge having length r ($r \geq 0$) in the Euclidean plane $\mathbf{R}^2$. In this paper, we determine $H_*(M_{n,r}; \mathbf{Z})$.

1. INTRODUCTION

We consider the configuration space $M_{n,r}$ of quasi-equilateral polygon linkages with n vertices, each edge having length 1 except for one fixed edge having length r ($r \geq 0$) in the Euclidean plane $\mathbf{R}^2$. More precisely, we define $\mathcal{C}_{n,r}$ ($n \geq 1$) by

$$\begin{aligned}(1.1) \qquad \mathcal{C}_{n,r} = \{(u_1, \ldots, u_n) \in (\mathbf{R}^2)^n : |u_{i+1} - u_i| = 1 \ \ (2 \leq i \leq n-1), \\ |u_1 - u_n| = 1, \ \text{and} \ |u_2 - u_1| = r\}.\end{aligned}$$

Note that $Iso^+(\mathbf{R}^2)$ (= the orientation preserving isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $SO(2)$) naturally acts on $\mathcal{C}_{n,r}$. We define $M_{n,r}$ by

$$(1.2) \qquad M_{n,r} = \mathcal{C}_{n,r}/Iso^+(\mathbf{R}^2) \ \text{for} \ r > 0, \ \text{and} \ M_{n,0} = \mathcal{C}_{n,0}/\mathbf{R}^2.$$

Then it is clear that $M_{n,r}$ ($r \geq 0$) can be described as follows:

$$(1.3) \qquad M_{n,r} = \Big\{(u_1, \ldots, u_n) \in \mathcal{C}_{n,r} : u_1 = (\frac{r}{2}, 0), u_2 = (-\frac{r}{2}, 0)\Big\}.$$

As $M_{n,n-1} = \{1\text{-point}\}$, and $M_{n,r} = \emptyset$ ($r > n-1$), we can assume that $r < n-1$. Concerning $M_{n,1}$, we have the following examples:

(i) $M_{3,1} = \{2\text{-points}\}$.

(ii) It is easy to see that $M_{4,1}$ is homeomorphic to $\{(x,y) \in \mathbf{R}^2 : (x+1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbf{R}^2 : (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 4\}$.

(iii) It is well known that $M_{5,1}$ is diffeomorphic to Σ_4, i.e., the compact, connected, and orientable two dimensional manifold of genus 4 (see for example [2], [3], [4]).

The dimension and the smoothness of $M_{n,r}$ are studied in [5] (cf. Proposition 2.3), where the Euler characteristics of $M_{n,r}$ are also determined. Finally, $M_{5,r}$ is treated in [11].

The purpose of this paper is to determine the homology group of $M_{n,r}$. The results are as follows.

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Theorem A.
(i) *For* $r \notin \mathbf{N}$, $M_{2m+1,r}$ *is homeomorphic to* M_{2m+1,k_1}, *where* k_1 *is the odd number which satisfies* $r-1 < k_1 < r+1$.
(ii) *For* $r \notin \mathbf{N}$, $M_{2m,r}$ *is homeomorphic to* M_{2m,k_2}, *where* k_2 *is the even number which satisfies* $r-1 < k_2 < r+1$.

By Theorem A, we need to know only $H_*(M_{n,k};\mathbf{Z})$, where $n \geq 3$ and $k \in \mathbf{N} \cup \{0\}$, in order to determine $H_*(M_{n,r};\mathbf{Z})$. Concerning $H_*(M_{n,k};\mathbf{Z})$, first we have the following:

Theorem B. $H_*(M_{n,k};\mathbf{Z})$ *is a torsion free module.*

Thus in order to determine $H_*(M_{n,k};\mathbf{Z})$, we need to describe the Poincaré polynomial $PS(n,k)$ of $M_{n,k}$, i.e., $PS(n,k) = \sum_\lambda \operatorname{rank} H_\lambda(M_{n,k};\mathbf{Z})t^\lambda$. Actually they are determined in Theorem 6.8. In particular, we have the following result for $k=1$.

Theorem C. *We have*

$$\text{(i) } PS(2m+1,1) = \sum_{\lambda=0}^{m-2} \binom{2m}{\lambda} t^\lambda + 2\binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-2} \binom{2m}{\lambda+2} t^\lambda \quad (m \geq 0).$$

$$\text{(ii) } PS(2m,1) = \sum_{\lambda=0}^{m-2} \binom{2m-1}{\lambda} t^\lambda + \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda \quad (m \geq 1),$$

where $\binom{a}{b}$ *is the binomial coefficient.*

Theorem C suggests that the Lefschetz hyperplane section theorem and the (partial) Poincaré duality might hold for $M_{n,1}$.

We recall the Lefschetz hyperplane section theorem (see [8]). Let V be a smooth algebraic variety of complex dimension l in $\mathbf{C}P^N$. Let P be a hyperplane in $\mathbf{C}P^N$. Then the maps $H_q(V \cap P;\mathbf{Z}) \to H_q(V;\mathbf{Z})$ induced from the inclusion are isomorphisms for $q \leq l-2$ and an epimorphism for $q = l-1$.

By setting $z_i = u_{i+2} - u_{i+1}$ $(1 \leq i \leq n-2)$, $z_{n-1} = u_1 - u_n$, and identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can write $M_{n,1}$ as

$$M_{n,1} \cong \{(z_1,\dots,z_{n-1}) \in (S^1)^{n-1} : z_1 + \dots + z_{n-1} - 1 = 0\}.$$

If we regard $(S^1)^{n-1}$ as a "variety" and $\{(z_1,\dots,z_{n-1}) \in (\mathbf{C})^{n-1} : z_1 + \dots + z_{n-1} - 1 = 0\}$ as a "hyperplane", then Theorem C might be indicating some Lefschetz-type theorem for $M_{n,1}$.

Concerning the (partial) Poincaré duality, as $M_{2m+1,1}$ is a smooth manifold of dimension $2m-2$ (cf. Proposition 2.3), Theorem C implies that $M_{2m+1,1}$ is orientable, and hence satisfies the Poincaré duality. On the other hand, $M_{2m,1}$ is a manifold of dimension $2m-3$ with isolated singular points. However, we have $H_{2m-3}(M_{2m,1};\mathbf{Z}) \cong \mathbf{Z}$. If we choose the fundamental class $[M_{2m,1}]$, it seems that the Poincaré duality homomorphisms $\bigcap[M_{2m,1}] : H^q(M_{2m,1};\mathbf{Z}) \to H_{2m-3-q}(M_{2m,1};\mathbf{Z})$ are isomorphisms for $q \leq m-3$ or $q \geq m$. Such situation occurs if $M_{2m,1}$ is an orientable $(m-2)$-regular space in the sense of [7].

In fact, we can prove these facts in a different way. We will give their proofs in a subsequent paper. We note that we can prove Theorem C from the two theorems together with the Euler characteristics $\chi(M_{n,1})$.

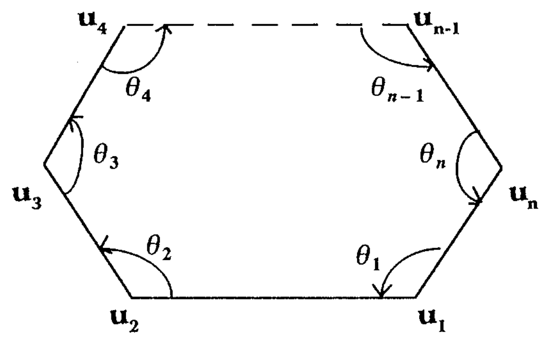


FIGURE 1

This paper is organized as follows. In §2, we define a map $\pi_{n,r} : M_{n,r} \to S^1$ for $r > 0$, and study the singular fibers of it. Then we prove Theorem A. Next we prepare some notations which are used in later sections.

In §3, we construct deformations $f^k_{n,r} : M_{n,k} \to M_{n,r}$ for $n - k \equiv 0$ *mod* 2 and $r \in (k-1, k+1)$ with $k \in \mathbf{N} \cup \{0\}$. This $f^k_{n,r}$ has limits $\lim_{r \to k-1+0} f^k_{n,r} : M_{n,k} \to M_{n,k-1}$ and $\lim_{r \to k+1-0} f^k_{n,r} : M_{n,k} \to M_{n,k+1}$. We denote the former limit by $f^-_{n,k}$, and the latter limit by $f^+_{n,k}$. We prove this in Theorem 3.1.

In §4, we first define the homotopy colimit of a diagram. Theorem 3.1 asserts that $M_{n,k}$ is actually described in the language of homotopy colimit. We prove this in Theorem 4.6. At the same time, using this description, we prove in Theorem 4.7 that $f^{\pm}_{n,k}$ can be decomposed into two maps if $n - k$ is even.

In §5, we prove exact sequences in homology, which are the key steps for calculating $H_*(M_{n,k})$ by induction. Theorems 5.6 and 5.6° are the main theorems of the section.

In §6, we first prove the properties which $f^{\pm}_{n,k} : H_*(M_{n,k}) \to H_*(M_{n,k\pm 1})$ satisfy, where $f^{\pm}_{n,k}$ are defined in §3. We do this in Theorem 6.3.

Having proved Theorem 6.3, we can apply Theorems 5.6 and 5.6° to $H_*(M_{n,k})$ and we establish recurrence relations for $PS(n, k)$. We do this in Theorem 6.7. Finally we solve the recurrence relations in Theorem 6.8. In particular, we have Theorems B and C.

2. NOTATIONS AND PRELIMINARIES

In this section we define a map $\pi_{n,r} : M_{n,r} \to S^1$ $(r > 0)$ and prove that it is a fiber bundle with singular fibers. Then we prepare some notations which are used in later sections.

For an element $(u_1, \ldots, u_n) \in M_{n,r}$ $(r \geq 0)$, we write coordinates of u_i as follows: As in (1.3), we set $u_1 = (\frac{r}{2}, 0)$ and $u_2 = (-\frac{r}{2}, 0)$. On the other hand, we set $u_i = (x_i, y_i)$ for $3 \leq i \leq n$. When $r > 0$, we also parameterize $M_{n,r}$ by parameters $\{(\theta_1, \ldots, \theta_n)\}$, where θ_i denotes the counter-clockwise angle from $\overrightarrow{u_i u_{i-1}}$ to $\overrightarrow{u_i u_{i+1}}$. (See Figure 1.)

We define $\pi_{n,r} : M_{n,r} \to S^1$ $(r > 0)$ by $\pi_{n,r}(\theta_1, \ldots, \theta_n) = \theta_2$. In order to study the singular fibers of $\pi_{n,r}$, we first consider the case of $r \in \mathbf{N}$. For such r, there exists a unique $\phi_r \in (0, \pi)$ so that if $(u_1, \ldots, u_n) \in \pi^{-1}_{n,r}(\phi_r)$, then $|u_3 - u_1| = r$. For later convenience, we write ϕ_r as ψ_r when r is odd, and ω_r when r is even. Then the structure of $\pi_{n,r}$ is given by the following:

Proposition 2.1.

(I) *The case* $n = 2m+1$:

(i) *The singular fibers of* $\pi_{2m+1,2i+1}$ *are given by* $\pi^{-1}_{2m+1,2i+1}(\psi_{2i+1})$ *and* $\pi^{-1}_{2m+1,2i+1}(-\psi_{2i+1})(0 \le i \le m-2)$. *Hence we have homeomorphisms:*

$$\pi^{-1}_{2m+1,2i+1}((-\psi_{2i+1}, \psi_{2i+1})) \cong (-\psi_{2i+1}, \psi_{2i+1}) \times M_{2m,2i},$$

$$\pi^{-1}_{2m+1,2i+1}((\psi_{2i+1}, 2\pi - \psi_{2i+1})) \cong (\psi_{2i+1}, 2\pi - \psi_{2i+1}) \times M_{2m,2i+2}.$$

(In what follows, we give only the singular fibers and omit such homeomorphisms.)

(ii) *The singular fibers of* $\pi_{2m+1,2m-1}$ *are given by* $\pi^{-1}_{2m+1,2m-1}(\psi_{2m-1})$ *and* $\pi^{-1}_{2m+1,2m-1}(-\psi_{2m-1})$.

(iii) *The singular fibers of* $\pi_{2m+1,2i}$ *are given by* $\pi^{-1}_{2m+1,2i}(0)$ *and* $\pi^{-1}_{2m+1,2i}(\pi)$ $(1 \le i \le m-1)$.

(II) *The case* $n = 2m$:

(iv) *The singular fibers of* $\pi_{2m,2i}$ *are given by* $\pi^{-1}_{2m,2i}(\omega_{2i})$ *and* $\pi^{-1}_{2m,2i}(-\omega_{2i})$ $(1 \le i \le m-2)$.

(v) *The singular fibers of* $\pi_{2m,2m-2}$ *are given by* $\pi^{-1}_{2m,2m-2}(\omega_{2m-2})$ *and* $\pi^{-1}_{2m,2m-2}(-\omega_{2m-2})$.

(vi) *The singular fibers of* $\pi_{2m,2i+1}$ *are given by* $\pi^{-1}_{2m,2i+1}(0)$ *and* $\pi^{-1}_{2m,2i+1}(\pi)$ $(1 \le i \le m-2)$.

Next we study the singular fibers of $\pi_{n,r}$ for $r \notin \mathbf{N}$. To do so, we generalize the definition of ψ_{2i+1} and ω_{2i} to ψ_r and ω_r as follows. For $r \notin \mathbf{N}$, we define $\psi_r \in (0, \pi)$ to satisfy the following property: If $(u_1, \dots, u_n) \in \pi^{-1}_{n,r}(\psi_r)$, then $|u_3 - u_1|$ is odd. (Of course it is the unique odd number which is contained in $(r-1, r+1)$.) Similarly for $r \notin \mathbf{N}$, we define $\omega_r \in (0, \pi)$ to satisfy the following property: If $(u_1, \dots, u_n) \in \pi^{-1}_{n,r}(\omega_r)$, then $|u_3 - u_1|$ is even.

Then we have the following proposition for $r \notin \mathbf{N}$.

Proposition 2.2.

(i) *The singular fibers of* $\pi_{2m+1,r}$ *are given by* $\pi^{-1}_{2m+1,r}(\psi_r)$ *and* $\pi^{-1}_{2m+1,r}(-\psi_r)$.

(ii) *The singular fibers of* $\pi_{2m,r}$ *are given by* $\pi^{-1}_{2m,r}(\omega_r)$ *and* $\pi^{-1}_{2m,r}(-\omega_r)$.

The proofs of Propositions 2.1 and 2.2 are elementary. So we indicate only the proof of Proposition 2.1. We define $f : M_{n,r} \to \mathbf{R}$ by $f(u_1, \dots, u_n) = |u_3 - u_1|^2$. We need to find the critical points of f. In order to do so, we recall the result concerning the smoothness of $M_{n,r}$ $(r > 0)$. In the following proposition, to say "x is a singular point of $M_{n,r}$" means that the Jacobian matrix of polynomial functions over $\mathbf{R}$, whose locus of common zeros is $M_{n,r}$, is not of maximal rank at x.

Proposition 2.3 ([2], [5])**.**

(i) $M_{2m+1,r}$ *is a smooth manifold of dimension* $2m-2$ *except for the case of* $r = 2i$ $(1 \le i \le m-1)$, *in which case* $(u_1, \dots, u_{2m+1})$ *is a singular point iff all of the* u_i *lie on the* x*-axis, i.e., the line determined by* u_1 *and* u_2.

(ii) $M_{2m,r}$ *is a smooth manifold of dimension* $2m-3$ *except for the case of* $r = 2i+1$ $(0 \le i \le m-2)$, *in which case* $(u_1, \dots, u_{2m})$ *is a singular point iff all of the* u_i *lie on the* x*-axis.*

Now the critical points of f are given by the following lemma:

Lemma 2.4. (i) $(u_1, \ldots, u_{2m+1}) \in M_{2m+1,2i+1}$ *is a critical point of* f *iff* $f(u_1, \ldots, u_{2m+1}) = (2i)^2, (2i+1)^2,$ *or* $(2i+2)^2$.

(ii) *A smooth point of* $(u_1, \ldots, u_{2m+1}) \in M_{2m+1,2i}$ *is a critical point of* f *iff* $f(u_1, \ldots, u_{2m+1}) = (2i-1)^2$ *or* $(2i+1)^2$.

(iii) $(u_1, \ldots, u_{2m}) \in M_{2m,2i}$ *is a critical point of* f *iff* $f(u_1, \ldots, u_{2m}) = (2i-1)^2, (2i)^2,$ *or* $(2i+1)^2$.

(iv) *A smooth point of* $(u_1, \ldots, u_{2m}) \in M_{2m,2i+1}$ *is a critical point of* f *iff* $f(u_1, \ldots, u_{2m+1}) = (2i)^2$ *or* $(2i+2)^2$.

We see that Proposition 2.1 follows easily from the facts in Morse theory applied to Lemma 2.4.

Proof of Lemma 2.4. We define

$$f_1(x_1, y_1, \ldots, x_n, y_n) = (x_1 - x_2)^2 + (y_1 - y_2)^2 - r^2,$$
$$f_i(x_1, y_1, \ldots, x_n, y_n) = (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 - 1 \ (2 \le i \le n-1),$$
$$f_n(x_1, y_1, \ldots, x_n, y_n) = (x_n - x_1)^2 + (y_n - y_1)^2 - 1.$$

By [9, Proposition 2.7], we see that a smooth point $(a_1, b_1, \ldots, a_n, b_n) \in \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbf{R}^{2n} : f_i(x_1, y_1, \ldots, x_n, y_n) = 0 \ (1 \le i \le n)\}$ is a critical point of f iff grad f is spanned by $\{\text{grad } f_1, \ldots, \text{grad } f_n\}$ at this point. By using this fact, we can prove Lemma 2.4 easily. □

Remark 2.5. As

$$\pi_{2m+1,2m-1}(M_{2m+1,2m-1}) = [-\psi_{2m-1}, \psi_{2m-1}]$$

and

$$\pi_{2m,2m-2}(M_{2m,2m-2}) = [-\omega_{2m-2}, \omega_{2m-2}],$$

the facts in Morse theory applied to Lemma 2.4 (i) and (iii) tell us that $M_{n,n-2}$ is homeomorphic to S^{n-3} $(n \ge 4)$.

Remark 2.6. We note that Proposition 2.3 supports the truth of Propositions 2.1 and 2.2. In fact, for example, we consider the case of $\pi_{2m+1,2i+1} : M_{2m+1,2i+1} \to S^1$. For each $\alpha \in S^1$, think of $\pi^{-1}_{2m+1,2i+1}(\alpha)$ as an element of $M_{2m,s}$ by corresponding $(u_1, \ldots, u_{2m+1}) \in \pi^{-1}_{2m+1,2i+1}(\alpha)$ to $(u_1, u_3, \ldots, u_{2m+1}) \in M_{2m,s}$, where we regard the line $\overline{u_1 u_3}$ as the fixed line, and we set $s = |u_3 - u_1|$.

If $\alpha \ne \pm\psi_{2i+1}$, then Proposition 2.3 tells us that $\pi^{-1}_{2m+1,2i+1}(\alpha)$ is a smooth manifold. On the other hand, if $\alpha = \pm\psi_{2i+1}$, then $\pi^{-1}_{2m+1,2i+1}(\alpha)$ has singular points. These facts indicate the truth of Propositions 2.1 and 2.2.

Proof of Theorem A. By applying Proposition 2.1 to $M_{2m+1,2i+1}$, and identifying $\pi^{-1}_{2m+1,2i+1}(\alpha)$ with $M_{2m,s}$ as in Remark 2.6, we have homeomorphisms $M_{2m,2i} \to M_{2m,s}$ for $2i \le s < 2i+1$. Similarly we have homeomorphisms $M_{2m,2i} \to M_{2m,s}$ for $2i-1 < s \le 2i$. Hence we have homeomorphisms $M_{2m,2i} \to M_{2m,s}$ for $2i-1 < s < 2i+1$.

Similarly we have homeomorphisms $M_{2m+1,2i+1} \to M_{2m+1,s}$ for $2i < s < 2i+2$. □

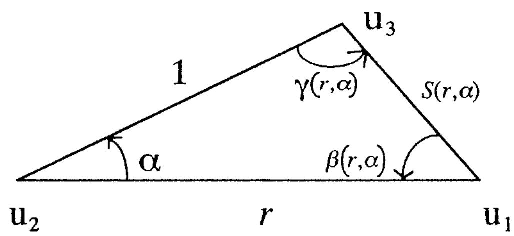


FIGURE 2

In §3, we construct the homeomorphisms in the proof of Theorem A more explicitly. In order to do so, we need some more notations, which we define in the rest of this section.

(1) Recall that we have a map $\pi_{n,r} : M_{n,r} \to S^1 (r > 0)$ given by $\pi_{n,r}(\theta_1, \ldots, \theta_n) = \theta_2$. As usual, we parametrize S^1 by the parameter α of counter-clockwise angle. For $\alpha \in S^1$, take an element $(\theta_1, \alpha, \theta_3, \ldots, \theta_n) \in \pi_{n,r}^{-1}(\alpha)$ (thus we write θ_2 as α), and think of the triangle with vertices (u_1, u_2, u_3). If $u_1 \neq u_3$, then we define $\beta(r, \alpha), \gamma(r, \alpha)$ and $s(r, \alpha)$ as follows.

$\beta(r, \alpha)$: the counter-clockwise angle from $\overrightarrow{u_1 u_3}$ to $\overrightarrow{u_1 u_2}$.
$\gamma(r, \alpha)$: the counter-clockwise angle from $\overrightarrow{u_3 u_2}$ to $\overrightarrow{u_3 u_1}$.
$s(r, \alpha)$: the distance $|u_3 - u_1|$.
(See Figure 2.)

Concerning $\beta(r, \alpha)$ and $\gamma(r, \alpha)$, we can easily prove the following properties, which are used in later sections.

Lemma 2.7.

(i) *$\beta(r, \alpha)$ and $\gamma(r, \alpha)$ are continuous except for the point where $(r, \alpha) = (1, 0)$. Moreover we have the following:*
  (a) *For $r > 1$, we have $\lim_{\alpha \to 0} \beta(r, \alpha) = 0$ and $\lim_{\alpha \to 0} \gamma(r, \alpha) = \pi$.*
  (b) *For $r < 1$, we have $\lim_{\alpha \to 0} \beta(r, \alpha) = \pi$ and $\lim_{\alpha \to 0} \gamma(r, \alpha) = 0$.*

(ii) *$\beta(r, \alpha)$ and $\gamma(r, \alpha)$ are not continuous at $(r, \alpha) = (1, 0)$. In fact we have the following:*
  (c) $\lim_{\alpha \to +0} \beta(1, \alpha) = \lim_{\alpha \to +0} \gamma(1, \alpha) = \frac{\pi}{2}$,
  (d) $\lim_{\alpha \to -0} \beta(1, \alpha) = \lim_{\alpha \to -0} \gamma(1, \alpha) = \frac{3\pi}{2}$.

(2) As in Remark 2.6, we can identify $\pi_{n,r}^{-1}(\alpha)$ with $M_{n-1, s(r,\alpha)}$ if $(r, \alpha) \neq (1, 0)$, i.e., $s(r, \alpha) \neq 0$. We write down this identification $\mu(r, \alpha) : \pi_{n,r}^{-1}(\alpha) \xrightarrow{\simeq} M_{n-1, s(r,\alpha)}$ more explicitly. Take $(\theta_1, \theta_2, \ldots, \theta_n) \in \pi_{n,r}^{-1}(\alpha)$. By (1), the triangle (u_1, u_2, u_3) can be written as $(\beta(r, \alpha), \alpha, \gamma(r, \alpha))$. Hence the $(n-1)$-gon $(u_1, u_3, u_4, \ldots, u_n)$ can be written as $(\theta_1 - \beta(r, \alpha), \theta_3 - \gamma(r, \alpha), \theta_4, \ldots, \theta_n)$. Let us write this element as $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_n)$. As $|u_3 - u_1| = s(r, \alpha)$, we can regard $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_n)$ as an element of $M_{n-1, s(r,\alpha)}$, with the fixed line $\overline{u_1 u_3}$. Hence we can define a homeomorphism $\mu(r, \alpha)$ by

$$\mu(r, \alpha)(\theta_1, \alpha, \theta_3, \theta_4, \ldots, \theta_n) = (\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_n). \tag{2.8}$$

(3) So far we have treated only $M_{n,r}$ with $r > 0$. Concerning $M_{n,0}$, we have the following:

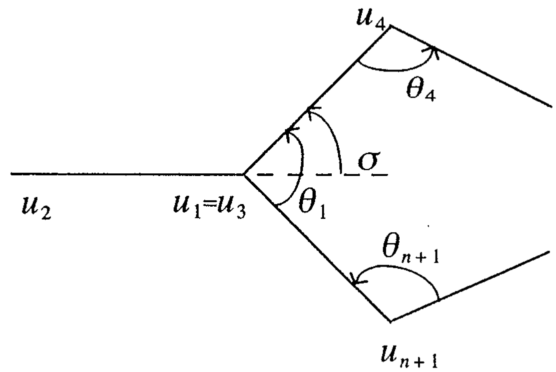


FIGURE 3

Lemma 2.9. *We have homeomorphisms*

$$\mu(1,0) : \pi^{-1}_{n+1,1}(0) \to M_{n,0},$$
$$\eta : M_{n,0} \to S^1 \times M_{n-1,1}.$$

Proof. For $(u_1, u_2, \ldots, u_{n+1}) \in \pi^{-1}_{n+1,1}(0)$, we set $\mu(1,0)(u_1, u_2, \ldots, u_{n+1}) = (u_1, u_3, u_4, \ldots, u_{n+1})$. As $u_1 = u_3$, we can assume that $(u_1, u_3, u_4, \ldots, u_{n+1}) \in M_{n,0}$.

For $(u_1, u_2, \ldots, u_n) \in M_{n,0}$ (recall that $u_1 = u_2 = O$ for such an element), we define $\pi_{n,0} : M_{n,0} \to S^1$ by $\pi_{n,0}((u_1, \ldots, u_n)) = u_3$. We identify u_3 with its counter-clockwise angle σ from $\overrightarrow{Oe_1}$ to $\overrightarrow{Ou_3}$, where we set $e_1 = \binom{1}{0}$.

If we form $(u_1, u_3, \ldots, u_n)$ from $(u_1, u_2, \ldots, u_n)$, then we can assume that $(u_1, u_3, \ldots, u_n) \in M_{n-1,1}$ with the fixed line $\overline{u_1u_3}$. (A more explicit identification is as follows: By rotating $(u_1, u_2, \ldots, u_n)$ by $\pi - \sigma$ around O, we have an element $(O, O, -e_1, u'_4, \ldots, u'_n)$. If we form $(\frac{1}{2}e_1, -\frac{1}{2}e_1, u'_4 + \frac{1}{2}e_1, \ldots, u'_n + \frac{1}{2}e_1)$, we can regard it as an element of $M_{n-1,1}$.)

Then we define η by $\eta(u_1, u_2, \ldots, u_n) = (\sigma, (\frac{1}{2}e_1, -\frac{1}{2}e_1, u'_4 + \frac{1}{2}e_1, \ldots, u'_n + \frac{1}{2}e_1))$. □

Remark 2.10. We note that $\mu(1,0)^{-1} \cdot \eta^{-1} : S^1 \times M_{n-1,1} \to \pi^{-1}_{n+1,1}(0)$ is given as follows: We write an element of $S^1 \times M_{n-1,1}$ as $(\sigma, (\theta_1, \theta_4, \ldots, \theta_{n+1}))$. Then

$$\mu(1,0)^{-1} \cdot \eta^{-1}(\sigma, (\theta_1, \theta_4, \ldots, \theta_{n+1})) = (\pi - \sigma + \theta_1, 0, \pi + \sigma, \theta_4, \ldots, \theta_{n+1}). \tag{2.11}$$

(See Figure 3.)

3. DEFORMATIONS OF POLYGONS

As indicated in §2, the purpose of this section is to prove the following:

Theorem 3.1. (I) *For $m \geq 3, 1 \leq k \leq n-2$ and $n - k \equiv 0$ mod 2, we have a map*

$$f^k_n : M_{n,k} \times (k-1, k+1) \to (S^1)^{n-1}$$

which satisfies the following properties (i)-(iv).

For $r \in (k-1, k+1)$, we define $f^k_{n,r} : M_{n,k} \to (S^1)^{n-1}$ to be the restriction of f^k_n on $M_{n,k} \times \{r\}$.

(i) $f^k_{n,r}$ *is injective and* Im $f^k_{n,r} = M_{n,r}$. *Hence* $f^k_{n,r} : M_{n,k} \to M_{n,r}$ *is a homeomorphism.*
(ii) $f^k_{n,k} = id$.
(iii) $\lim_{r\to k-1+0} f^k_{n,r}$ *and* $\lim_{r\to k+1-0} f^k_{n,r}$ *exist. We set*

$$f^-_{n,k} = \lim_{r\to k-1+0} f^k_{n,r} \quad \text{and} \quad f^+_{n,k} = \lim_{r\to k+1-0} f^k_{n,r}.$$

(iv) *By (iii), we see that* $f^k_n : M_{n,k} \times (k-1, k+1) \to (S^1)^{n-1}$ *is extendable to a map* $f^k_n : M_{n,k} \times [k-1, k+1] \to (S^1)^{n-1}$. *We require that the latter map is continuous.*

(II) *Moreover, when* $n \equiv 0$ *mod* 2, *we have a map*

$$f^0_n : M_{n,0} \times [0,1) \to (S^1)^{n-1}$$

which satisfies properties similar to those in (I).

For the rest of this section, we prove Theorem 3.1. To avoid confusion, we set

$$\begin{cases} f^{2i+1}_{2m+1} = \tau^{2i+1}_{2m+1} : M_{2m+1,2i+1} \times (2i, 2i+2) \to (S^1)^{2m}, & k = 2i+1, \\ f^{2i}_{2m} = \rho^{2i}_{2m} : M_{2m,2i} \times (2i-1, 2i+1) \to (S^1)^{2m-1}, & k = 2i. \end{cases}$$

When the number of vertices $n = 2m+1$ or $n = 2m$ is clearly understood, we drop these indices from τ^{2i+1}_{2m+1} or ρ^{2i}_{2m}. Thus $f^k_{n,r} : M_{n,k} \to M_{n,r}$ in Theorem 3.1 is written as

$$\begin{cases} \tau^{2i+1}_r : M_{2m+1,2i+1} \to M_{2m+1,r}, \quad r \in (2i, 2i+2), & k = 2i+1, \\ \rho^{2i}_r : M_{2m,2i} \to M_{2m,r}, \quad r \in (2i-1, 2i+1), & k = 2i. \end{cases}$$

Before beginning the proof, we explain the essential idea for constructing τ^{2i+1}_r and ρ^{2i}_r. As their ideas are similar, we explain for τ^{2i+1}_r. Take an element $(\theta_1, \theta_2, \dots, \theta_{2m+1}) \in M_{2m+1,2i+1}$. As in §2 (2), we separate it into a triangle $(\beta(2i+1, \alpha), \alpha, \gamma(2i+1, \alpha))$ and a $2m$-gon $(\bar\theta_1, \bar\theta_3, \theta_4, \dots, \theta_{2m+1})$, where $\bar\theta_1 = \theta_1 - \beta(2i+1, \alpha), \bar\theta_3 = \theta_3 - \gamma(2i+1, \alpha)$.

(i) First deform the triangle to $(\beta(r, \alpha'), \alpha', \gamma(r, \alpha'))$, i.e., a triangle with the length of the fixed line being r (α' is chosen suitably). Note that the length of the oblique side of this new triangle is equal to $s(r, \alpha')$.

(ii) Next think of $(\bar\theta_1, \bar\theta_3, \theta_4, \dots, \theta_{2m+1}) \in \pi^{-1}_{2m+1,2i+1}(\alpha)$ as an element of $M_{2m,s(2i+1,\alpha)}$ by $\mu(r, \alpha)$ (cf. §2 (2)). Then deform $(\bar\theta_1, \bar\theta_3, \theta_4, \dots, \theta_{2m+1})$ to $(\bar\theta'_1, \bar\theta'_3, \theta'_4, \dots, \theta'_{2m+1})$, so that the length of the fixed line of this new $2m$-gon, i.e., $\overline{u_1u_3}$, is equal to $s(r, \alpha')$.

(iii) Finally attach $(\beta(r, \alpha'), \alpha', \gamma(r, \alpha'))$ and $(\bar\theta'_1, \bar\theta'_3, \theta'_4, \dots, \theta'_{2m+1})$ along the lines of length $s(r, \alpha')$, i.e., $\overline{u_1u_3}$. Then we get a new $(2m+1)$-gon

$$(\bar\theta'_1 + \beta(r, \alpha'), \alpha', \bar\theta'_3 + \gamma(r, \alpha'), \theta'_4, \dots, \theta'_{2m+1}). \tag{3.2}$$

We denote this $(2m+1)$-gon as $\tau^{2i+1}_r(\theta_1, \theta_2, \dots, \theta_{2m+1})$.

Hereafter, we use the following notations.

Notation 3.3. (i) If τ^{2i+1}_r is constructed as in Theorem 3.1 (I), then we define $\tau^{r_1}_{r_2} : M_{2m+1,r_1} \to M_{2m+1,r_2}$ $(2i < r_1, r_2 < 2i+2)$ by $\tau^{r_1}_{r_2} = \tau^{2i+1}_{r_2} \cdot (\tau^{2i+1}_{r_1})^{-1}$.

(ii) If ρ^0_r or ρ^{2i}_r is constructed as in Theorem 3.1 (I) or (II), then we define $\rho^{r_1}_{r_2} : M_{2m,r_1} \to M_{2m,r_2}$ $(0 \le r_1, r_2 < 1)$ or $\rho^{r_1}_{r_2} : M_{2m,r_1} \to M_{2m,r_2}$ $(2i-1 < r_1, r_2 < 2i+1)$ in the same way as we defined $\tau^{r_1}_{r_2}$.

Now we prove Theorem 3.1 by induction on n, where n is the number of vertices of polygons, i.e., $n = 2m+1$ or $2m$. For the initial condition of the induction, it is clear that we have a family of homeomorphisms $\tau^1 : M_{3,1} \times (0,2) \to (S^1)^2$.

(A) Assume that $\tau^{2i+1} : M_{2m+1,2i+1} \times (2i, 2i+2) \to (S^1)^{2m}$ are constructed for $0 \le i \le m-1$.

We need to construct $\rho^0 : M_{2m+2,0} \times [0,1) \to (S^1)^{2m+1}$, $\rho^2 : M_{2m+2,2} \times (1,3) \to (S^1)^{2m+1}$, and $\rho^{2i} : M_{2m+2,2i} \times (2i-1, 2i+1) \to (S^1)^{2m+1}$ for $2 \le i \le m$. In order to do so, it suffices to construct

$$\rho_r^0 : M_{2m+2,0} \to M_{2m+2,r}, \quad r \in [0,1),$$
$$\rho_r^2 : M_{2m+2,2} \to M_{2m+2,r}, \quad r \in (1,3),$$

and

$$\rho_r^{2i} : M_{2m+2,2i} \to M_{2m+2,r}, \quad r \in (2i-1, 2i+1).$$

(i) Constructions of ρ_r^{2i} for $2 \le i \le m$. First we deform $(\beta(2i,\alpha), \alpha, \gamma(2i,\alpha))$. In order to do so, we only need to designate how α changes as r moves, i.e. to designate a function $g_r^{2i} : [0, 2\pi] \to [0, 2\pi]$. We consider the properties which g_r^{2i} should satisfy.

As ρ_r^{2i} should satisfy $\rho_{2i}^{2i} = id$, g_r^{2i} should satisfy

$$g_{2i}^{2i} = id. \tag{3.4}$$

Actually g_r^{2i} should satisfy one more property. Recall that $\pi_{2m+2,r}^{-1}(\omega_r)$ is a singular fiber for $2i-1 < r < 2i+1$ by Propositions 2.1 and 2.2. Think of the situation that u_1 moves with u_2 fixed, i.e., the length r of the fixed line moves away from $2i$. In this situation, the singular fiber $\pi_{2m+2,2i}^{-1}(\omega_{2i})$ should be deformed to a singular fiber $\pi_{2m+2,r}^{-1}(\omega_r)$. Equivalently, in the course of deformation of the triangle $(\beta(2i,\omega_{2i}), \omega_{2i}, \gamma(2i,\omega_{2i}))$, the length of the oblique side ($= s(r, g_r^{2i}(\omega_{2i}))$) should always satisfy $s(r, g_r^{2i}(\omega_{2i})) = 2i$. And, by the definition of ω_r, this is equivalent to

$$g_r^{2i}(\omega_{2i}) = \omega_r \;\; (2i-1 < r < 2i+1). \tag{3.5}$$

Now it is natural that we define g_r^{2i}, which satisfies (3.4) and (3.5), by the following manner: Think of a graph ω_r in the $\{r\} \times \{\alpha\}$ plane. If r moves in $(2i-1, 2i+1)$, then ω_r is a decreasing function, so

$$\lim_{r \to 2i-1+0} \omega_r = \pi, \qquad \lim_{r \to 2i+1-0} \omega_r = 0.$$

An element $\alpha \in [0,\pi]$ can be written as $\alpha = \lambda\omega_{2i}$ or $\alpha = \lambda\omega_{2i} + (1-\lambda)\pi$ $(0 \le \lambda \le 1)$. So we define $g_r^{2i}(\alpha)$ to be the internal dividing point of $[0, \omega_r]$ or $[\omega_r, \pi]$ which preserves λ, i.e., for $\alpha \in [0,\pi]$,

$$g_r^{2i}(\alpha) = \begin{cases} \lambda\omega_r & \text{if } \alpha = \lambda\omega_{2i} \;\; (0 \le \lambda \le 1), \\ \lambda\omega_r + (1-\lambda)\pi & \text{if } \alpha = \lambda\omega_{2i} + (1-\lambda)\pi \;\; (0 \le \lambda \le 1). \end{cases} \tag{3.6}$$

Finally we define $g_r^{2i}(-\alpha)$ for $\alpha \in [0,\pi]$ by

$$g_r^{2i}(-\alpha) = -g_r^{2i}(\alpha) \quad \text{for } \alpha \in [0,\pi]. \tag{3.7}$$

Thus we have completed the definition of g_r^{2i}.

Now we deform a triangle $(\beta(2i,\alpha), \alpha, \gamma(2i,\alpha))$ to the triangle

$$(\beta(r, g_r^{2i}(\alpha)), g_r^{2i}(\alpha), \gamma(r, g_r^{2i}(\alpha))).$$

Next we deform $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2})$. Note that $|u_3 - u_1|$ of this $(2m+1)$-gon is equal to $s(2i, \alpha)$. On the other hand, the oblique side of the deformed triangle has length $s(r, g_r^{2i}(\alpha))$. So we need to deform $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2})$ so that the length of the fixed line $(= |u_1 - u_3|)$ is equal to $s(r, g_r^{2i}(\alpha))$. Note that by the definition of g_r^{2i}, we have

(a) If $2i - 1 \le s(2i, \alpha) < 2i$, then $2i - 1 \le s(r, g_r^{2i}(\alpha)) < 2i$.
(b) If $2i < s(2i, \alpha) \le 2i + 1$, then $2i < s(r, g_r^{2i}(\alpha)) \le 2i + 1$.

By the inductive hypothesis, we have a homeomorphism $\tau_{r_2}^{r_1} : M_{2m+1,r_1} \to M_{2m+1,r_2}$ for $2i - 1 \le r_1, r_2 < 2i$ or $2i < r_1, r_2 \le 2i + 1$. Hence if $\alpha \ne \pm\omega_{2i}$, we can deform $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2})$ to

$$\tau^{s(2i,\alpha)}_{s(r,g_r^{2i}(\alpha))}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2}) \in M_{2m+1,s(r,g_r^{2i}(\alpha))}. \tag{3.8}$$

If $\alpha = \pm\omega_{2i}$, we define the deformation of $M_{2m+1,2i}$ to itself by the identity map.

Finally we attach the deformed triangle and the deformed $(2m+1)$-gon along the lines of length $s(r, g_r^{2i}(\alpha))$, i.e., $\overline{u_1 u_3}$. Then we have the following definition of ρ_r^{2i}.

$$\begin{aligned} &\rho_r^{2i}(\theta_1, \alpha, \theta_3, \ldots, \theta_{2m+2}) \\ &= \begin{cases} (\beta(r, g_r^{2i}(\alpha)), g_r^{2i}(\alpha), \gamma(r, g_r^{2i}(\alpha))) \dot{+} \tau^{s(2i,\alpha)}_{s(r,g_r^{2i}(\alpha))}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2}) \\ \qquad\qquad\qquad \text{for } \alpha \ne \pm\omega_{2i}, \\ (\beta(r, g_r^{2i}(\alpha)), g_r^{2i}(\alpha), \gamma(r, g_r^{2i}(\alpha))) \dot{+} (\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2}) \\ \qquad\qquad\qquad \text{for } \alpha = \pm\omega_{2i}, \end{cases} \end{aligned} \tag{3.9}$$

where the symbol $\dot{+}$ is defined as follows: For example, for

$$(\beta(r, g_r^{2i}(\alpha)), g_r^{2i}(\alpha), \gamma(r, g_r^{2i}(\alpha))) \dot{+} \tau^{s(2i,\alpha)}_{s(r,g_r^{2i}(\alpha))}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2}),$$

write $\tau^{s(2i,\alpha)}_{s(r,g_r^{2i}(\alpha))}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+2})$ as $(\bar{\theta}'_1, \bar{\theta}'_3, \theta'_4, \ldots, \theta'_{2m+2})$. Then the above formula is understood as $(\bar{\theta}'_1 + \beta(r, g_r^{2i}(\alpha)), g_r^{2i}(\alpha), \bar{\theta}'_3 + \gamma(r, g_r^{2i}(\alpha)), \theta'_4, \ldots, \theta'_{2m+2})$.

In the following we always use the symbol $\dot{+}$ in this sense.

Concerning (3.9), if we fix $r \in (2i-1, 2i+1)$, then ρ_r^{2i} given in (3.9) is continuous at $\pi^{-1}_{2m+2,r}(\pm\omega_r)$, because of the inductive hypothesis of Theorem 3.1 (I). By using Lemma 2.7 (i), we can easily prove that ρ_r^{2i} satisfies the conditions of Theorem 3.1 (I). This completes the constructions of ρ_r^{2i} $(2i - 1 < r < 2i + 1)$ for $2 \le i \le m$.

(ii) Construction of ρ_r^2. In order to construct ρ_r^2 $(1 < r < 3)$, note that ρ_r^2 $(2 \le r < 3)$ can be constructed in the same way as in (i). So we need to construct ρ_r^2 $(1 < r \le 2)$. If we follow the steps in (i) in order to construct ρ_r^2 $(1 < r \le 2)$, we see that this deformation is not continuous as $r \to 1$ and $\alpha \to 0$. The essential reason for this is that $\beta(r, \alpha)$ and $\gamma(r, \alpha)$ are not continuous at $(r, \alpha) = (1, 0)$ in the notation of Lemma 2.7. Hence we replace g_r^{2i} in (i) by a nicer function G_r^2, i.e., deform triangles $(\beta(2, \alpha), \alpha, \gamma(2, \alpha))$ more nicely.

For the same reason as in (3.4) and (3.5), we must have $G_2^2 = id$ and $G_r^2(\omega_2) = \omega_r$. Think of a graph ω_r in the $\{r\} \times \{\alpha\}$ plane. If r moves in $(1, 2]$, then ω_r is a decreasing function, so that $\lim_{r \to 1+0} \omega_r = \pi$. Choose a number, say $\frac{\pi}{3}$ which satisfies $0 < \frac{\pi}{3} < \omega_2$, then think of a line l_0 in the $\{r\} \times \{\alpha\}$ plane, which contains the coordinates $(1, 0)$ and $(2, \frac{\pi}{3})$. We see that l_0 contains $(r, \frac{\pi}{3}(r - 1))$. An element $\alpha \in [0, \pi]$ can be written as $\alpha = \lambda\frac{\pi}{3}$, $\alpha = \lambda\frac{\pi}{3} + (1 - \lambda)\omega_2$, or $\alpha = \lambda\omega_2 + (1 - \lambda)\pi$ $(0 \le$

$\lambda \leq 1$). So let $G_r^2(\alpha)$ be the internal dividing point of $[0, \frac{\pi}{3}(r-1)], [\frac{\pi}{3}(r-1), \omega_r]$, or $[\omega_r, \pi]$, which preserves λ, i.e., for $\alpha \in [0, \pi]$,

$$\text{(3.10)} \quad G_r^2(\alpha) = \begin{cases} \lambda\frac{\pi}{3}(r-1) & \text{if } \alpha = \lambda\frac{\pi}{3} \ (0 \leq \lambda \leq 1), \\ \lambda\frac{\pi}{3}(r-1) + (1-\lambda)\omega_r & \text{if } \alpha = \lambda\frac{\pi}{3} + (1-\lambda)\omega_2 \ (0 \leq \lambda \leq 1), \\ \lambda\omega_r + (1-\lambda)\pi & \text{if } \alpha = \lambda\omega_2 + (1-\lambda)\pi \ (0 \leq \lambda \leq 1). \end{cases}$$

Then define $G_r^2(-\alpha) = -G_r^2(\alpha)$ for $\alpha \in [0, \pi]$.

Finally we define ρ_r^2 as in (3.9), i.e.,

$$\text{(3.11)} \quad \rho_r^2(\theta_1, \alpha, \theta_3, \ldots, \theta_{2m+2}) = \begin{cases} (\beta(r, G_r^2(\alpha)), G_r^2(\alpha), \gamma(r, G_r^2(\alpha))) \\ \qquad \dot{+}\tau^{s(2,\alpha)}_{s(r,G_r^2(\alpha))}(\bar\theta_1, \bar\theta_3, \theta_4, \ldots, \theta_{2m+2}) & \text{for } \alpha \neq \pm\omega_2, \\ (\beta(r, G_r^2(\alpha)), G_r^2(\alpha), \gamma(r, G_r^2(\alpha)))\dot{+}(\bar\theta_1, \bar\theta_3, \theta_4, \ldots, \theta_{2m+2}) & \text{for } \alpha = \pm\omega_2. \end{cases}$$

The fact that ρ_r^2 is continuous as $r \to 1$ and $\alpha \to 0$ can be checked easily by using the following fact, which tells us that $(\beta(r, G_r^2(\alpha)), G_r^2(\alpha), \gamma(r, G_r^2(\alpha)))$ is continuous as $r \to 1$ and $\alpha \to 0$: Assume that $|\alpha|$ is small. Then we can write $\alpha = \lambda\frac{\pi}{3}$, and $G_r^2(\alpha) = \lambda\frac{\pi}{3}(r-1)$. From the deformed triangle $(\beta(r, G_r^2(\alpha)), G_r^2(\alpha), \gamma(r, G_r^2(\alpha)))$, we know that

$$\sin^2\beta(r, G_r^2(\alpha)) = \frac{\sin^2\lambda\frac{\pi}{3}(r-1)}{r^2 + 1 - 2r\cos\lambda\frac{\pi}{3}(r-1)}.$$

Then we can prove that

$$\lim_{r\to 1}\sin^2\beta(r, G_r^2(\alpha)) = \frac{\lambda^2\pi^2}{9 + \lambda^2\pi^2}.$$

This completes the construction of ρ_r^2 $(1 < r < 3)$.

Remark 3.12. Note that when $|\alpha|$ is small, we have constructed G_r^2 by the idea of a real version of "blowing-up."

(iii) Construction of ρ_r^0. In order to construct ρ_r^0 $(0 \leq r < 1)$, first we construct ρ_r^0 $(0 \leq r \leq \frac{1}{2})$. Take an element of $M_{2m+2,0}$ and by η (cf. Lemma 2.9), regard it as an element of $S^1 \times M_{2m+1,1}$ and write it as $(\sigma, (\theta_1, \theta_3, \ldots, \theta_{2m+2}))$. Prepare a triangle $(\beta(r,\sigma), \sigma, \gamma(r,\sigma))$. (Note that the length of the oblique side of this triangle is equal to $s(r,\sigma)$.) Deform $(\theta_1, \theta_3, \ldots, \theta_{2m+2}) \in M_{2m+1,1}$ to $\tau^1_{s(r,\alpha)}(\theta_1, \theta_3, \ldots, \theta_{2m+2})$. Finally attach this triangle and $(2m+1)$-gon along the lines of length $s(r,\alpha)$. Thus

$$\text{(3.13)} \quad \rho_r^0(\sigma, (\theta_1, \theta_3, \ldots, \theta_{2m+2})) = (\beta(r,\sigma), \sigma, \gamma(r,\sigma))\dot{+}\tau^1_{s(r,\alpha)}(\theta_1, \theta_3, \ldots, \theta_{2m+2}),$$

for $0 \leq r \leq \frac{1}{2}$.

Next we define the deformation $\rho_r^{\frac{1}{2}} : M_{2m+2,\frac{1}{2}} \to M_{2m+2,r}$ $(\frac{1}{2} \leq r < 1)$ in the same way as in ρ_r^2 $(1 < r \leq 2)$, i.e., by "blowing-up" along the line through $(1,0)$ and $(\frac{1}{2}, \frac{\pi}{3})$. Then define ρ_r^0 $(\frac{1}{2} \leq r < 1)$ by $\rho_r^0 = \rho_r^{\frac{1}{2}} \cdot \rho^0_{\frac{1}{2}}$. This completes the construction of ρ_r^0 $(0 \leq r < 1)$.

(B) Assume $\rho^0 : M_{2m,0} \times [0,1) \to (S^1)^{2m-1}$ and $\rho^{2i} : M_{2m,2i} \times (2i-1, 2i+1) \to (S^1)^{2m-1}$ are constructed for $1 \leq i \leq m-1$.

We need to construct $\tau^{2i+1} : M_{2m+1,2i+1} \times (2i, 2i+2) \to (S^1)^{2m}$ for $0 \le i \le m-1$. In order to do so, it suffices to construct

$$\tau_r^{2i+1} : M_{2m+1,2i+1} \to M_{2m+1,r}, \quad r \in (2i, 2i+2).$$

We note that τ_r^{2i+1} $(1 \le i \le m-1)$ can be constructed in the same way as in (A)-(i) by taking ψ_r instead of ω_r (as for ψ_r, see Propositions 2.1 and 2.2). Thus we need to construct only τ_r^1 $(0 < r < 2)$. In order to do so, we can construct τ_r^1 independently for the two cases $0 < r \le 1$ and $1 \le r < 2$. But as the constructions for these two cases are similar, we construct τ_r^1 only for $1 \le r < 2$.

By the same reason as in the construction of (A)-(ii), the essential part for which we must be careful in order to construct τ_r^1 is the part for $\tau_{1+\epsilon}^1$ (where $\epsilon > 0$ is small), i.e., a deformation of an element of $M_{2m+1,1}$ to a $(2m+1)$-gon whose fixed line has length slightly larger than 1. We construct τ_r^1 in the following way.

Fix a number r $(1 \le r < 2)$, and think of $\epsilon > 0$ as a variable. We construct a homeomorphism $\tau_{1+\epsilon}^r : M_{2m+1,r} \to M_{2m+1,1+\epsilon}$ such that $\lim_{\epsilon \to +0} \tau_{1+\epsilon}^r$ exists and defines a homeomorphism. We write this limit as τ_1^r. Finally define τ_r^1, which we need to define, as $(\tau_1^r)^{-1}$. (Note that τ_1^r is a homeomorphism.)

Since constructions of $\tau_{1+\epsilon}^r$ are similar for a fixed r, we set $r = \frac{3}{2}$ in order to make sure that r is fixed. In order to construct $\tau_{1+\epsilon}^{\frac{3}{2}} : M_{2m+1,\frac{3}{2}} \to M_{2m+1,1+\epsilon}$, we deform the triangle $(\beta(\frac{3}{2},\alpha), \alpha, \gamma(\frac{3}{2},\alpha))$ in the same way as in (A)-(i), i.e., define $h_{1+\epsilon}^{\frac{3}{2}}(\alpha)$ by

$$h_{1+\epsilon}^{\frac{2}{3}}(\alpha) = \begin{cases} \lambda\psi_{1+\epsilon} & \text{if } \alpha = \lambda\psi_{\frac{3}{2}} \ (0 \le \lambda \le 1), \\ \lambda\psi_{1+\epsilon} + (1-\lambda)\pi & \text{if } \alpha = \lambda\psi_{\frac{3}{2}} + (1-\lambda)\pi \ (0 \le \lambda \le 1). \end{cases}$$

(Recall that $s(1+\epsilon, \psi_{1+\epsilon}) = 1$.) Then the deformed triangle is given by

$$(\beta(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)), h_{1+\epsilon}^{\frac{3}{2}}(\alpha), \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))).$$

We note that $\beta(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))$ is not continuous as $\epsilon \to +0$ and $\alpha \to 0$ by Lemma 2.7. In order to make $\tau_{1+\epsilon}^{\frac{3}{2}}$ continuous, we need to deform $\pi_{2m+1,\frac{3}{2}}^{-1}(\alpha)$ suitably, so that [the deformed triangle] $\dot{+}$ [the deformed $2m$-gon] is continuous, although [the deformed triangle] itself is not continuous as above.

When $|\alpha|$ is small, we deform $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1}) \in \pi_{2m+1,\frac{3}{2}}^{-1}(\alpha)$ by the following idea: As usual, we assume that $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1}) \in M_{2m,s(\frac{3}{2},\alpha)}$ by $\mu(\frac{3}{2},\alpha)$. Write $\rho_{s(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))}^{s(\frac{3}{2},\alpha)}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1})$ as $(\bar{\theta}'_1, \bar{\theta}'_3, \theta'_4, \ldots, \theta'_{2m+1})$. If $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1})$ can be deformed to

$$(\bar{\theta}''_1, \bar{\theta}''_3, \theta''_4, \ldots, \theta''_{2m+1}) \in M_{2m, s(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))}$$

so that

$$\begin{aligned} &\bar{\theta}''_1 \approx \bar{\theta}'_1 - \{\gamma(\frac{3}{2},\alpha) - \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))\}, \\ &\bar{\theta}''_3 \approx \bar{\theta}'_3 + \{\gamma(\frac{3}{2},\alpha) - \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))\}, \\ &\theta''_4 \approx \theta'_4, \ldots, \theta''_{2m+1} \approx \theta'_{2m+1} \end{aligned} \tag{3.14}$$

($\approx$ means approximately the same), then we can define $\tau_{1+\epsilon}^{\frac{3}{2}}(\theta_1, \alpha, \theta_3, \ldots, \theta_{2m+1})$ by

$$\tau_{1+\epsilon}^{\frac{3}{2}}(\theta_1, \alpha, \theta_3, \ldots, \theta_{2m+1}) = (\beta(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)), h_{1+\epsilon}^{\frac{3}{2}}(\alpha), \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))) \dot{+} (\bar{\theta}_1'', \bar{\theta}_3'', \theta_4'', \ldots, \theta_{2m+1}''). \tag{3.15}$$

In fact, by (3.14) we have

$$\tau_{1+\epsilon}^{\frac{3}{2}}(\theta_1, \alpha, \theta_3, \ldots, \theta_{2m+1}) \approx (\bar{\theta}_1' + \beta(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)) + \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)) - \gamma(\frac{3}{2}, \alpha), h_{1+\epsilon}^{\frac{3}{2}}(\alpha), \bar{\theta}_3' + \gamma(\frac{3}{2}, \alpha), \theta_4', \ldots, \theta_{2m+1}'). \tag{3.16}$$

We know that

$$\lim_{\substack{\alpha \to 0 \\ r \to 1}} \beta(r, \alpha) + \gamma(r, \alpha) = \pi$$

in the notation of Lemma 2.7. Hence the term

$$\beta(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)) + \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))$$

in (3.16) is continuous as $\epsilon \to +0$ and $\alpha \to 0$. Thus $\tau_{1+\epsilon}^{\frac{3}{2}}$ is continuous.

For small $|\alpha|$, the deformation of $(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1})$ to $(\bar{\theta}_1'', \bar{\theta}_3'', \theta_4'', \ldots, \theta_{2m+1}'')$ is defined as follows.

(a) First get an element $\rho_0^{s(\frac{3}{2},\alpha)}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1}) \in M_{2m,0}$, and identify $M_{2m,0}$ with $S^1 \times M_{2m-1,1}$ by η (cf. Lemma 2.9.) So we can write the above element as $(\sigma, (\Theta_1, \Theta_4, \Theta_5, \ldots, \Theta_{2m+1}))$, where $\sigma, \Theta_1, \Theta_4, \Theta_5, \ldots, \Theta_{2m+1} \in [0, 2\pi]$.

(b) Next form $(\sigma + \gamma(\frac{3}{2}, \alpha) - \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)), (\Theta_1, \Theta_4, \ldots, \Theta_{2m+1}))$, i.e., rotate $(\sigma, (\Theta_1, \Theta_4, \ldots, \Theta_{2m+1}))$ by $\gamma(\frac{3}{2}, \alpha) - \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))$ in the counter-clockwise angle.

(c) Finally get an element

$$\rho^0_{s(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))}(\sigma + \gamma(\frac{3}{2}, \alpha) - \gamma(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha)), (\Theta_1, \Theta_4, \ldots, \Theta_{2m+1})) \in M_{2m, s(1+\epsilon, h_{1+\epsilon}^{\frac{3}{2}}(\alpha))}.$$

(By using the definition of ρ_r^0, it is easily checked that the steps (a), (b) and (c) satisfy (3.14).)

Then we finally need to extend this $\tau_{1+\epsilon}^{\frac{3}{2}}$ continuously for any α. (As usual we need to deform a singular fiber $\pi_{2m+1,\frac{3}{2}}^{-1}(\psi_{\frac{3}{2}})$ to $\pi_{2m+1,1+\epsilon}^{-1}(\psi_{1+\epsilon})$.)

Thus it is natural to define $\tau_{1+\epsilon}^{\frac{3}{2}}$ in the following manner: Choose a small positive number, say $\frac{1}{100}$, then choose a continuous function $F : [0, \pi] \to [0, 1]$ so that $F(\alpha) = 1$ for $0 \le \alpha \le \psi_{\frac{3}{2}} - \frac{1}{100}$, and $F(\alpha) = 0$ for $\psi_{\frac{3}{2}} \le \alpha \le \pi$. Then extend the domain of F to $[0, 2\pi]$ by setting $F(-\alpha) = F(\alpha)$ for $\alpha \in [0, \pi]$.

Take an element $(\theta_1, \alpha, \theta_3, \ldots, \theta_{2m+1}) \in M_{2m+1,\frac{3}{2}}$.

(1) If $s(\frac{3}{2}, \alpha) < 1$, then write $\rho_0^{s(\frac{3}{2},\alpha)}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \ldots, \theta_{2m+1})$ as

$$(\sigma, (\Theta_1, \Theta_4, \ldots, \Theta_{2m+1})).$$

And set

(3.17)
$$\tau^{\frac{3}{2}}_{1+\epsilon}(\theta_1, \alpha, \theta_3, \dots, \theta_{2m+1}) = (\beta(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha)), h^{\frac{3}{2}}_{1+\epsilon}(\alpha), \gamma(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha)))$$
$$\dot{+}\rho^0_{s(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha))}(\sigma + F(\alpha)\{\gamma(\frac{3}{2}, \alpha) - \gamma(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha))\}, (\Theta_1, \Theta_4, \dots, \Theta_{2m+1})).$$

(2) If $s(\frac{3}{2}, \alpha) > 1$, then set

$$\tau^{\frac{3}{2}}_{1+\epsilon}(\theta_1, \alpha, \theta_3, \dots, \theta_{2m+1}) = (\beta(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha)), h^{\frac{3}{2}}_{1+\epsilon}(\alpha), \gamma(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha))) \dot{+}\rho^{s(\frac{3}{2},\alpha)}_{s(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha))}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \dots, \theta_{2m+1}). \tag{3.18}$$

(3) If $s(\frac{3}{2}, \alpha) = 1$, then set

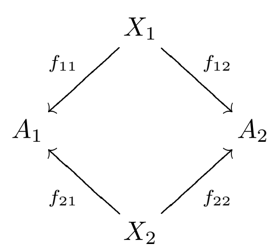
$$\tau^{\frac{3}{2}}_{1+\epsilon}(\theta_1, \alpha, \theta_3, \dots, \theta_{2m+1}) = (\beta(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha)), h^{\frac{3}{2}}_{1+\epsilon}(\alpha), \gamma(1+\epsilon, h^{\frac{3}{2}}_{1+\epsilon}(\alpha))) \dot{+}(\bar{\theta}_1, \bar{\theta}_3, \theta_4, \dots, \theta_{2m+1}). \tag{3.19}$$

We can easily prove that $\lim_{\epsilon \to +0} \tau^{\frac{3}{2}}_{1+\epsilon}$ exists and defines a homeomorphism. This completes the construction of τ^1_r $(0 < r < 2)$, and consequently, the proof of Theorem 3.1.

4. Models for $M_{n,k}$

In this section, we construct a space which has the same homotopy type as $M_{n,k}$, where $k \in \mathbf{Z}$. We prepare some notations.

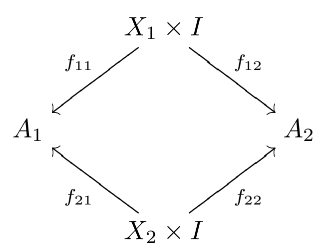
We consider the following diagram:



where X_1, X_2, A_1, A_2 are spaces and f_{ij} are continuous maps. To this diagram, we correspond a space, which is called the homotopy colimit [1], as follows. Homotopy colimit is a quotient space obtained from the topological sum of $X_1 \times I, X_2 \times I, A_1$ and A_2 by identifying $(x_i, -1) \in X_i \times I$ with $f_{i1}(x_i) \in A_1$ and $(x_i, 1) \in X_i \times I$ with $f_{i2}(x_i) \in A_2$ for $i = 1, 2$, where $I = [-1, 1]$.

Let us write this homotopy colimit by

(4.1)



Let $\mathcal{H}\mathsf{o}$ be the homotopy category. (For two topological spaces X and Y, $X = Y$ in $\mathcal{H}\mathsf{o}$ means that X and Y are homotopy equivalent. And for $f, g : X \to Y$, $f = g$ means that f is homotopic to g.)

As for the homotopy colimit, the following lemma is well known.

Lemma 4.2. *The homotopy colimit is a homotopy functor (from the category of diagrams to the homotopy category). Thus the homotopy colimit depends only on the homotopy equivalences of X_i and the homotopy classes of f_{ij}.*

In the following Definitions 4.3 and 4.3°, we consider special diagrams.

Definition 4.3. We consider the following diagram:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{s} & & \searrow^{s} & \\ A & & & & A \\ & \nwarrow_{t} & & \nearrow_{t} & \\ & & Y & & \end{array}$$

We denote its homotopy colimit by $I(s, t)$. In particular, if $Y = A$ and $t = id_A : A \to A$, then we denote $I(s, id_A)$ by $J(s)$.

Definition 4.3°. We consider the following diagram:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{p} & & \searrow^{q} & \\ A & & & & B \\ & \nwarrow_{p} & & \nearrow_{q} & \\ & & X & & \end{array}$$

We denote its homotopy colimit by $II(p, q)$. In particular, if $X = B$ and $q = id_B : B \to B$, then we denote $II(p, \mathrm{id}_B)$ by $J'(p)$.

Lemma 4.4. *In $\mathcal{H}\mathsf{o}$, we have $J(s) = J'(s)$ for $s : X \to A$.*

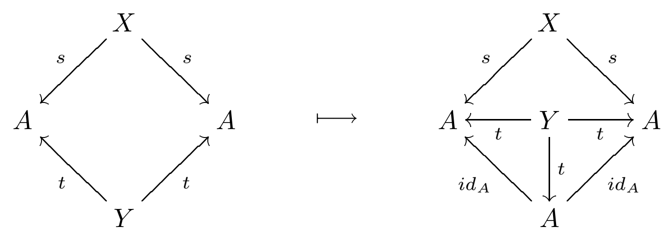
Proof. The definitions of $J(s)$ and $J'(s)$ are as follows:

$$\begin{array}{ccccc} & & X \times I & & \\ & \swarrow^{s} & & \searrow^{s} & \\ J(s) = A & & & & A \\ & \nwarrow_{id_A} & & \nearrow_{id_A} & \\ & & A \times I & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & X \times I & & \\ & \swarrow^{s} & & \searrow^{id_x} & \\ J'(s) = A & & & & X \\ & \nwarrow_{s} & & \nearrow_{id_X} & \\ & & X \times I & & \end{array}$$

We define a map $J(s) \to J'(s)$. Let $c : I \to I$ be the scalar change defined by $c(x) = 2x + 1,\ -1 \leq x \leq 0$ and $c(x) = -2x + 1,\ 0 \leq x \leq 1$. Then the map is defined by sending $A \times I$ in $J(s)$ to A in $J'(s)$ by the projection to the first factor, and then sending $X \times I$ in $J(s)$ to $X \times I \cup X \times I$ in $J'(s)$ via $id_X \times c$. We see that the map induces a homotopy equivalence. ☐

We will define the contraction maps which play the essential role to attain our purpose.

Definition 4.5. (i) To a diagram in Definition 4.3, we define its lower contraction as follows:

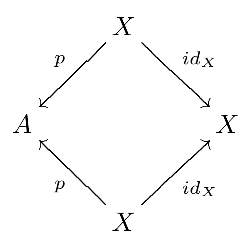


This morphism induces a map between homotopy colimits $\gamma_1(s) : I(s,t) \to J(s)$.

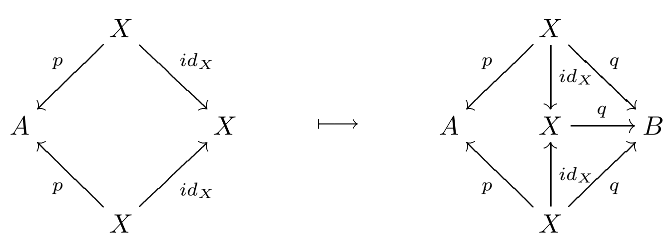
(ii) Similarly we can define the upper contraction map $\gamma_2(t) : I(s,t) \to J(t)$.

(iii) If we consider (ii) to the case when $Y = A$ and $t = id_A : A \to A$, then we have a map $\gamma_2(id_A) : I(s, id_A) \to J(id_A)$. Since $I(s, id_A) = J(s)$ and $J(id_A) \simeq S^1 \times A$, we can write this map as $\gamma_2 : J(s) \to S^1 \times A$.

Definition 4.5°. (i) For a diagram

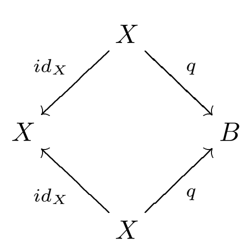


and a map $q : X \to B$, we define its right contraction as follows:



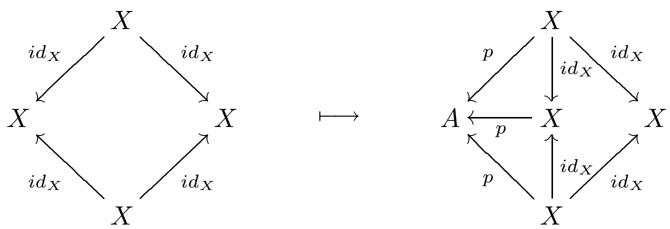
This morphism induces a map between homotopy colimits $\lambda_1(q) : J'(p) \to II(p,q)$.

(ii) Similarly, for a diagram



and a map $p : X \to A$, we can define the left contraction map $\lambda_2(p) : J'(q) \to II(p, q)$.

(iii) If we consider (ii) in the case when $B = X$ and $q = id_X : X \to X$, then we have a map $\lambda_2(p) : S^1 \times X \to J'(p)$, which is induced from the following map:



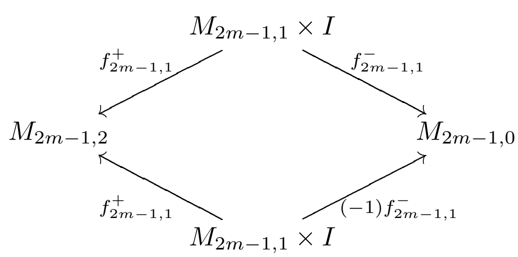
Theorem 4.6. *In $\mathcal{H}o$, we have*

$$M_{n,k} = \begin{cases} I(f^-_{n-1,k+1}, f^+_{n-1,k-1}) & \text{if } n-k \text{ is even and } k \geq 1, \\ II(f^+_{n-1,k}, f^-_{n-1,k}) & \text{if } n-k \text{ is odd and } k \geq 1. \end{cases}$$

We recall that $f^{\pm}_{n-1,k} : M_{n-1,k} \to M_{n-1,k\pm1}$ are defined in Theorem 3.1.

Proof. For $2 \leq k \leq n-3$, this theorem is the direct translation of Theorem 3.1. For $k \geq n-2$, this theorem is still valid if we remember that the empty set has the unique map $\emptyset$ into any space.

So there remains the case $M_{n,1}$. We only consider $M_{2m,1}$. We define a map $(-1) : M_{2m-1,0} \to M_{2m-1,0}$ as taking $(u_1, u_2, \ldots, u_{2m-1})$ to $(-u_1, -u_2, \ldots, -u_{2m-1})$, where we note that $u_1 = u_2 = 0$ in this case. By Theorem 3.1 (see the construction of $\tau^{\frac{3}{2}}_{1+\epsilon}$), $M_{2m,1}$ is homeomorphic to



Hence in order to complete the proof, it is enough to show that (-1) is homotopic to the identity by Lemma 4.2. If we define a homotopy $\overline{f_t} : S^1 \times M_{2m-2,1} \to S^1 \times M_{2m-2,1}$ by a rotation $\overline{f_t}(\alpha, u) = (e^{i\pi t}\alpha, u)$ with $0 \leq t \leq 1$, $\alpha \in S^1$ and $u \in M_{2m-2,1}$, then the induced homotopy $f_t = \eta^{-1} \cdot \overline{f_t} \cdot \eta : M_{2m-1,0} \to M_{2m-1,0}$ ($\eta : M_{2m-1,0} \xrightarrow{\simeq} S^1 \times M_{2m-2,1}$ is defined in Lemma 2.9) is the desired homotopy.

We can apply the same argument to $M_{2m+1,1}$. ∎

Theorem 4.7. *We have the following commutative diagrams in $\mathcal{H}o$.*

(A) (i) *For $n - k \equiv 0$ mod 2 and $k \geq 1$,*

$$\begin{array}{ccccc} M_{n,k} & & \xrightarrow{f^+_{n,k}} & & M_{n,k+1} \\ \| & & & & \| \\ I(f^-_{n-1,k+1}, f^+_{n-1,k-1}) & \xrightarrow{\gamma_1(f^-_{n-1,k+1})} & J(f^-_{n-1,k+1}) & \xrightarrow{\lambda_2(f^+_{n-1,k+1})} & II(f^+_{n-1,k+1}, f^-_{n-1,k+1}). \end{array}$$

(ii) *For $n-k \equiv 0 \bmod 2$ and $k \geq 2$,*

$$\begin{array}{ccccc} M_{n,k} & & \xrightarrow{f^-_{n,k}} & & M_{n,k-1} \\ \| & & & & \| \\ I(f^-_{n-1,k+1}, f^+_{n-1,k-1}) & \xrightarrow{\gamma_2(f^+_{n-1,k+1})} & J(f^+_{n-1,k-1}) & \xrightarrow{\lambda_1(f^-_{n-1,k-1})} & II(f^+_{n-1,k-1}, f^-_{n-1,k-1}). \end{array}$$

(B) (iii)

$$\begin{array}{ccccc} M_{2m,0} & & \xrightarrow{f^+_{2m,0}} & & M_{2m,1} \\ \| & & & & \| \\ S^1 \times M_{2m-1,1} & \xrightarrow{\lambda_2(f^+_{2m-1,1})} & J(f^+_{2m-1,1}) & \xrightarrow{\lambda_1(f^-_{2m-1,1})} & II(f^+_{2m-1,1}, f^-_{2m-1,1}). \end{array}$$

(iv)

$$\begin{array}{ccccc} M_{2m+1,1} & & \xrightarrow{f^-_{2m+1,1}} & & M_{2m+1,0} \\ \| & & & & \| \\ I(f^-_{2m,2}, f^+_{2m,1}) & \xrightarrow{\gamma_1(f^-_{2m,2})} & J(f^-_{2m,2}) & \xrightarrow{\gamma_2} & S^1 \times M_{2m,1}. \end{array}$$

Proof. We will prove A (i), as the other cases are handled in a similar way. Recall that $I(f^-_{n-1,k+1}, f^+_{n-1,k-1})$ and $II(f^+_{n-1,k+1}, f^-_{n-1,k+1})$ are defined by the following colimits:

$$\begin{array}{ccccc} & & M_{n-1,k+1} \times I & & \\ & \swarrow^{f^-_{n-1,k+1}} & & \searrow^{f^-_{n-1,k+1}} & \\ I(f^-_{n-1,k+1}, f^+_{n-1,k-1}) = M_{n-1,k} & & & & M_{n-1,k} \\ & \nwarrow_{f^+_{n-1,k-1}} & & \nearrow_{f^+_{n-1,k-1}} & \\ & & M_{n-1,k-1} \times I & & \end{array}$$

and

$$\begin{array}{ccccc} & & M_{n-1,k+1} \times I & & \\ & \swarrow^{f^+_{n-1,k+1}} & & \searrow^{f^-_{n-1,k+1}} & \\ II(f^+_{n-1,k+1}, f^-_{n-1,k+1}) = M_{n-1,k+2} & & & & M_{n-1,k} \\ & \nwarrow_{f^+_{n-1,k+1}} & & \nearrow_{f^-_{n-1,k+1}} & \\ & & M_{n-1,k+1} \times I & & \end{array}$$

where we recall that two copies of $M_{n-1,k}$ in $I(f^-_{n-1,k+1}, f^+_{n-1,k-1})$ correspond to the singular fibers of the projection $\pi_{n,k} : M_{n,k} \to S^1$ (see §2).

By the construction of $f^+_{n,k} : M_{n,k} \to M_{n,k+1}$, this map is characterized as follows.

(i) For $(u,t) \in M_{n-1,k-1} \times I$, we have

$$f^+_{n,k}(u,t) = f^+_{n-1,k-1}(u) \in II(f^+_{n-1,k+1}, f^-_{n-1,k+1}).$$

(ii) $f^+_{n,k}$ maps $M_{n-1,k+1} \times I$ to $M_{n-1,k+1} \times I \cup M_{n-1,k+1} \times I$ by a scalar change of I (see c in the proof of Lemma 4.4).

Next we look at $\lambda_2(f^-_{n-1,k+1})\gamma_1(f^-_{n-1,k+1})$.

(iii) Recall that $J(f^-_{n-1,k+1})$ is defined by the following colimit:

$$\begin{array}{ccccc} & & M_{n-1,k+1} \times I & & \\ & \swarrow^{f^-_{n-1,k+1}} & & \searrow^{f^-_{n-1,k+1}} & \\ J(f^-_{n-1,k+1}) = M_{n-1,k} & & & & M_{n-1,k} \\ & \nwarrow_{id} & & \nearrow_{id} & \\ & & M_{n-1,k} \times I & & \end{array}$$

It is easy to see that $J(f^-_{n-1,k+1})$ is homotopy equivalent to a quotient space obtained from the topological sum of $M_{n-1,k+1} \times I$ and $M_{n-1,k}$ by identifying $(u,0) \in M_{n-1,k+1} \times I$ with $f^-_{n-1,k+1}(u) \in M_{n-1,k}$, and $(u,1) \in M_{n-1,k+1} \times I$ with $f^-_{n-1,k+1}(u) \in M_{n-1,k}$. Hence by $\gamma_1(f^-_{n-1,k+1})$, $(u,t) \in M_{n-1,k-1} \times I$ is mapped to $f^+_{n-1,k-1}(u) \in M_{n-1,k}$.

(iv) By $\lambda_2(f^+_{n-1,k+1})$, $f^+_{n-1,k-1}(u) \in M_{n-1,k}$ (see (iii)) is mapped to itself in $II(f^+_{n-1,k+1}, f^-_{n-1,k+1})$.

Now by (i)-(ii) and (iii)-(iv), we see that $f^+_{n,k}$ and $\lambda_2(f^-_{n-1,k+1})\gamma_1(f^-_{n-1,k+1})$ are homotopic. □

5. Auxiliary theorems

We prepare some theorems which are useful to clarify the proof of Theorems B and C. We begin by proving two lemmas.

Lemma 5.1. *We consider the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & C & \longrightarrow & B & \xrightarrow{j_3} & A \\ & & \downarrow h & & \downarrow f & & \downarrow g & & \| \\ 0 & \longrightarrow & D' & \longrightarrow & C' & \longrightarrow & B' & \xrightarrow{j'_3} & A. \end{array}$$

Here the upper and lower sequences are exact and h is surjective. Then we have the following isomorphism (i) and an exact sequence (ii).

(i) $\mathrm{Coker}\ f \cong (\mathrm{Ker}\ j'_3 + \mathrm{Im}\ g)/\mathrm{Im}\ g$.

(ii) $0 \longrightarrow \mathrm{Ker}\ h \longrightarrow \mathrm{Ker}\ f \longrightarrow \mathrm{Ker}\ g \longrightarrow 0$.

Proof. We have the diagram of two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & C & \longrightarrow & \mathrm{Ker}\ j_3 & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow f & & \downarrow g|\mathrm{Ker}\ j_3 & & \\ 0 & \longrightarrow & D' & \longrightarrow & C' & \longrightarrow & \mathrm{Ker}\ j'_3 & \longrightarrow & 0. \end{array}$$

Hence we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker}\ h \longrightarrow \mathrm{Ker}\ f \longrightarrow \mathrm{Ker}\ (g|\mathrm{Ker}\ j_3) \longrightarrow \mathrm{Coker}\ h \longrightarrow \mathrm{Coker}\ f \\ \longrightarrow \mathrm{Coker}\ (g|\mathrm{Ker}\ j_3) \longrightarrow 0. \end{aligned}$$

Since h is surjective, we have isomorphisms

$$0 \longrightarrow \text{Coker } f \longrightarrow \text{Coker } (g|\text{Ker } j_3) \longrightarrow 0 \tag{5.2}$$

and

$$0 \longrightarrow \text{Ker } h \longrightarrow \text{Ker } f \longrightarrow \text{Ker } (g|\text{Ker } j_3) \longrightarrow 0.$$

From the definition, we have

$$\text{Coker } (g|\text{Ker } j_3) = \text{Ker } j_3'/(\text{Im } g \cap \text{Ker } j_3') \cong (\text{Ker } j_3' + \text{Im } g)/\text{Im } g.$$

Since $j_3 = j_3' g$, we see that $\text{Ker } g \subset \text{Ker } j_3$. Hence we get

$$\text{Ker } (g|\text{Ker } j_3) = \text{Ker } g \cap \text{Ker } j_3 = \text{Ker } g.$$

Now the lemma follows from (5.2) □

Dually we have

Lemma 5.3. *We consider the diagram*

$$\begin{array}{ccccccccc} A & \xrightarrow{i_1} & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0 \\ \| & & \downarrow g & & \downarrow f & & \downarrow h & & \\ A & \xrightarrow{i_1'} & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & 0. \end{array}$$

Here the upper and lower sequences are exact and h is injective. Then we have the following isomorphism (i) and an exact sequence (ii).

(i) $\text{Ker } f \cong \text{Ker } g/(\text{Ker } g \cap \text{Im } i_1)$.
(ii) $0 \longrightarrow \text{Coker } g \longrightarrow \text{Coker } f \longrightarrow \text{Coker } h \longrightarrow 0$.

Proof. We have the diagram of two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B/\text{Im } i_1 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0 \\ & & \downarrow \bar{g} & & \downarrow f & & \downarrow h & & \\ 0 & \longrightarrow & B'/\text{Im } i_1' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & 0, \end{array}$$

where $\bar{g}$ is the induced map of g. Hence we get the long exact sequence

$$0 \longrightarrow \text{Ker } \bar{g} \longrightarrow \text{Ker } f \longrightarrow \text{Ker } h \longrightarrow \text{Coker } \bar{g} \longrightarrow \text{Coker } f \longrightarrow \text{Coker } h \longrightarrow 0.$$

Since h is injective, we get

$$0 \longrightarrow \text{Ker } \bar{g} \longrightarrow \text{Ker } f \longrightarrow 0 \tag{5.4}$$

and

$$0 \longrightarrow \text{Coker } \bar{g} \longrightarrow \text{Coker } f \longrightarrow \text{Coker } h \longrightarrow 0.$$

From the definition, we have

$$\text{Ker } f \cong \text{Ker } \bar{g} = (\text{Ker } g + \text{Im } i_1)/\text{Im } i_1 \cong \text{Ker } g/(\text{Ker } g \cap \text{Im } i_1)$$

and

$$\text{Coker } \bar{g} = (B'/\text{Im } i_1')\,/\,((\text{Im } g + \text{Im } i_1')/\text{Im } i_1') \cong B'/(\text{Im } g + \text{Im } i_1').$$

Noting that $i_1' = g i_1$, we see that $\text{Im } i_1' \subset \text{Im } g$. It means that

$$\text{Coker } \bar{g} \cong B'/\text{Im } g = \text{Coker } g.$$

Now the lemma follows from (5.4). □

Notation 5.5. We collect notations here. Let K be any field. Then we denote the homology with K-coefficients by $H_*(X)$. We assume that $H_n(X) = 0$ for $X = \emptyset$ or $n < 0$. Furthermore we assume that the map $\emptyset$ from the empty set to any space induces $\emptyset_* = 0$ on homology groups.

Theorem 5.6. *For $s : X \to A$ and $t : Y \to A$, we assume that* $\mathrm{Im}\ (s_*) \subset \mathrm{Im}\ (t_*)$. *Then the following (i) and (ii) hold.*

(i) *We have the following isomorphism*:

$$H_j(I(s,t)) \cong H_j(A) \oplus \mathrm{Coker}_j\ (t_*) \oplus H_{j-1}(X) \oplus \mathrm{Ker}_{j-1}\ (t_*).$$

In particular, we have

$$H_j(J(t)) \cong H_j(A) \oplus H_{j-1}(Y).$$

As for $I(s,t)$ and $J(t)$, see §4.

(ii) *We have the following exact sequences*:

$$0 \longrightarrow \mathrm{Coker}_j\ (t_*) \oplus \mathrm{Ker}_{j-1}(t_*) \longrightarrow H_j(I(s,t)) \xrightarrow{\gamma_1(s)_*} H_j(J(s)) \longrightarrow 0$$

and

$$\begin{aligned} 0 \longrightarrow \mathrm{Coker}_j\ (t_*) \oplus \mathrm{Ker}_{j-1}(s_*) \longrightarrow H_j(I(s,t)) \xrightarrow{\gamma_2(t)_*} H_j(J(t)) \\ \longrightarrow \mathrm{Im}_{j-1}\ (t_*)/\mathrm{Im}_{j-1}\ (s_*) \longrightarrow 0. \end{aligned}$$

Proof. (i) From the Mayer-Vietoris sequence for $I(s,t)$, we have

$$\begin{aligned} \cdots \longrightarrow H_j(X) \oplus H_j(Y) \xrightarrow{\mathcal{M}_j} H_j(A) \oplus H_j(A) \xrightarrow{(\iota_+)_* + (\iota_-)_*} H_j(I(s,t)) \\ \xrightarrow{\partial_j} H_{j-1}(X) \oplus H_{j-1}(Y) \xrightarrow{\mathcal{M}_{j-1}} \cdots, \end{aligned}$$

where $\iota_\pm : A \to I(s,t)$ are the two inclusions and $\mathcal{M}_j$ is a map defined by

$$(x, y) \mapsto (s_*x - t_*y, s_*x - t_*y)$$

for $x \in H_j(X)$ and $y \in H_j(Y)$.

It is immediate that $\mathrm{Ker}\ \mathcal{M}_{j-1} \cong H_{j-1}(X) \oplus \mathrm{Ker}_{j-1}(t_*)$. More explicitly, let $\tau_{j-1} : H_{j-1}(X) \to H_{j-1}(Y)$ be a map satisfying $t_*\tau_{j-1}(x) = s_*(x)$. Then $\mathrm{Ker}\ \mathcal{M}_{j-1} \cong \tilde{\Delta} H_{j-1}(X) \oplus \mathrm{Ker}_{j-1}(t_*)$, where we set $\tilde{\Delta}(x)$ equal to $(x, \tau_{j-1}(x))$ for $x \in H_{j-1}(X)$.

Let $\Delta : A \to A \times A$ be the diagonal map. Then we see that $\mathrm{Im}\ \mathcal{M}_j = \Delta \mathrm{Im}_j\ (t_*)$. Furthermore if we write a section of the projection $H_j(A) \to \mathrm{Coker}_j(t_*)$ as $\tilde{\sigma}_j$, then we have the following decomposition as a vector space:

$$H_j(A) \oplus H_j(A) \cong H_j(A) \oplus \Delta \mathrm{Im}_j(t_*) \oplus \tilde{\sigma}_j \mathrm{Coker}_j\ (t_*),$$

where the first factor of the left-hand side corresponds to the first factor of the right-hand side. Hence we have

$$\mathrm{Coker}\ \mathcal{M}_j \cong H_j(A) \oplus \mathrm{Coker}_j\ (t_*).$$

Now we have the exact sequence

$$0 \longrightarrow H_j(A) \oplus \mathrm{Coker}_j(t_*) \longrightarrow H_j(I(s,t)) \longrightarrow \mathrm{Ker}_{j-1}\ (t_*) \oplus H_{j-1}(X) \longrightarrow 0.$$

This completes the proof of (i).

(ii) We will give a proof of the second exact sequence. We recall that $J(t)$ is defined by

$$\begin{array}{ccccc} & & Y \times I & & \\ & \overset{t}{\swarrow} & & \overset{t}{\searrow} & \\ J(t) = A & & & & A \\ & \underset{id_A}{\nwarrow} & & \underset{id_A}{\nearrow} & \\ & & A \times I & & \end{array}$$

Comparing the Mayer-Vietoris sequences for $I(s,t)$ and $J(t)$, we obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_j(A) \oplus \mathrm{Coker}_j(t_*) & \longrightarrow & H_j(I(s,t)) & \longrightarrow & H_{j-1}(X) \oplus H_{j-1}(Y) \\ & & \downarrow{\scriptstyle id+0} & & \downarrow{\scriptstyle \gamma_2(t)_*} & & \downarrow{\scriptstyle s_*+id} \\ 0 & \longrightarrow & H_j(A) & \longrightarrow & H_j(J(t)) & \longrightarrow & H_{j-1}(A) \oplus H_{j-1}(Y) \end{array}$$

$$\begin{array}{cc} \xrightarrow{\mathcal{M}_{j-1}} & H_{j-1}(A) \oplus H_{j-1}(A) \\ & \| \\ \xrightarrow{\mathcal{M}'_{j-1}} & H_{j-1}(A) \oplus H_{j-1}(A). \end{array}$$

Applying Lemma 5.1, we see that

$$\mathrm{Coker}_j\ (\gamma_2(t)_*) \cong \left(\mathrm{Ker}\ \mathcal{M}'_{j-1} +\ \mathrm{Im}_{j-1}\ (s_* + id\ )\right)/\mathrm{Im}_{j-1}\ (s_* + id\ )$$

and

$$0 \longrightarrow \mathrm{Ker}_j\ (id\ \ + 0) \longrightarrow \mathrm{Ker}_j\ (\gamma_2(t)_*) \longrightarrow \mathrm{Ker}_{j-1}\ (s_* + id\ ) \longrightarrow 0.$$

It is easily seen that Ker $\mathcal{M}'_{j-1} = \{(t_* y, y) : y \in H_{j-1}(Y)\} \cong H_{j-1}(Y)$. Since Im $(s_*) \subset$ Im (t_*), we see that Ker $\mathcal{M}'_{j-1}+\mathrm{Im}_{j-1}\ (s_*+id\ ) = \mathrm{Im}_{j-1}\ (t_*)\oplus H_{j-1}(Y)$. So we have $\mathrm{Coker}_j\ (\gamma_2(t)_*) \cong \mathrm{Im}_{j-1}\ (t_*)/\mathrm{Im}_{j-1}\ (s_*)$.

It is immediate that $\mathrm{Ker}_j\ (id\ \ + 0) = \mathrm{Coker}_j\ (t_*)$ and $\mathrm{Ker}_{j-1}(s_* + id\ ) = \mathrm{Ker}_{j-1}\ (s_*)$. Thus we have $\mathrm{Ker}_j\ (\gamma_2(t)_*) \cong \mathrm{Coker}_j\ (t_*) \oplus \mathrm{Ker}_{j-1}\ (s_*)$.

Now it is easy to prove (ii). □

The following theorem is the dual of Theorem 5.6.

Theorem 5.6°. *For $p : X \to A$ and $q : X \to B$, we assume that* Ker $(q_*) \subset$ Ker (p_*). *Then the following (i) and (ii) hold.*

(i) *We have the following isomorphism*:

$$H_j(II(p,q)) \cong H_j(A) \oplus \mathrm{Coker}_j\ (q_*) \oplus H_{j-1}(X) \oplus \mathrm{Ker}_{j-1}\ (q_*).$$

In particular, we have

$$H_j(J'(p)) \cong H_j(A) \oplus H_{j-1}(X).$$

As for $II(p,q)$ and $J'(p)$, see §4.

(ii) *We have the following exact sequences*:

$$0 \longrightarrow H_j(J'(p)) \xrightarrow{\lambda_1(q)_*} H_j(II(p,q)) \longrightarrow \mathrm{Coker}_j\ (q_*) \oplus \mathrm{Ker}_{j-1}\ (q_*) \longrightarrow 0$$

and

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker}_j\ (p_*)/\mathrm{Ker}_j\ (q_*) \longrightarrow H_j(J'(q)) \xrightarrow{\lambda_2(p)_*} H_j(II(p,q)) \\ \longrightarrow \mathrm{Coker}_j\ (p_*) \oplus \mathrm{Ker}_{j-1}\ (q_*) \longrightarrow 0. \end{aligned}$$

Proof. (i) From the Mayer-Vietoris sequence for $II(p,q)$, we have

$$\begin{aligned} \cdots \longrightarrow H_j(X) \oplus H_j(X) \xrightarrow{\mathcal{M}_j} H_j(A) \oplus H_j(B) \xrightarrow{(\iota_+)_* + (\iota_-)_*} H_j(II(p,q)) \\ \xrightarrow{\partial_j} H_{j-1}(X) \oplus H_{j-1}(X) \longrightarrow \cdots, \end{aligned}$$

where $\iota_+ : A \to II(p,q)$ and $\iota_- : B \to II(p,q)$ are the inclusions and $\mathcal{M}_j$ is a map defined by

$$(x, y) \mapsto (p_*x - p_*y, q_*x - q_*y)$$

for $(x,y) \in H_j(X) \oplus H_j(X)$.

Since Ker $(q_*) \subset$ Ker (p_*), we have Im $\partial_j \cong \mathrm{Ker}_{j-1}\ (q_*) \oplus \Delta H_{j-1}(X)$, where $\mathrm{Ker}_{j-1}\ (q_*)$ is a subspace of the first factor of $H_{j-1}(X) \oplus H_{j-1}(X)$ and $\Delta : H_{j-1}(X) \to H_{j-1}(X) \oplus H_{j-1}(X)$ is the diagonal map. So we see that

$$\mathrm{Coker}\ \mathcal{M}_j \cong (H_j(A) \oplus H_j(B))/\mathcal{M}_j\left((H_j(X) \oplus H_j(X))/\left(\mathrm{Ker}_j\ (q_*) \oplus \Delta H_j(X)\right)\right).$$

Using $H_j(X) \oplus H_j(X) \cong H_j(X) \oplus \Delta H_j(X)$, we have

$$(H_j(X) \oplus H_j(X))/\left(\mathrm{Ker}_j\ (q_*) \oplus \Delta H_j(X)\right) \cong H_j(X)/\mathrm{Ker}_j\ (q_*) \cong \mathrm{Im}_j\ (q_*).$$

Hence we get

$$\mathrm{Coker}\ \mathcal{M}_j \cong H_j(A) \oplus (H_j(B)/\mathrm{Im}_j\ (q_*)) \cong H_j(A) \oplus \mathrm{Coker}_j\ (q_*).$$

Now we have the following exact sequence:

$$0 \longrightarrow H_j(A) \oplus \mathrm{Coker}_j\ (q_*) \longrightarrow H_j(II(p,q)) \longrightarrow H_{j-1}(X) \oplus \mathrm{Ker}_{j-1}\ (q_*) \longrightarrow 0.$$

This completes the proof of (i).

(ii) We will give a proof of the second exact sequence. Comparing the Mayer-Vietoris sequences for $J'(q)$ and $II(p,q)$, we obtain

$$\begin{array}{ccccc} H_j(X) & \xrightarrow{\bar{\mathcal{M}}'_j} & H_j(X) \oplus H_j(B) & \longrightarrow & H_j(J'(q)) \\ \| & & \downarrow{\scriptstyle p_*+id} & & \downarrow{\scriptstyle \lambda_2(p)_*} \\ H_j(X) & \xrightarrow{\bar{\mathcal{M}}_j} & H_j(A) \oplus H_j(B) & \longrightarrow & H_j(II(p,q)) \\ & \longrightarrow & \Delta H_{j-1}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0+id} & & \\ & \longrightarrow & \mathrm{Ker}_{j-1}\ (q_*) \oplus \Delta H_{j-1}(X) & \longrightarrow & 0 \end{array}$$

where $\bar{\mathcal{M}}'_j(\alpha) = (\alpha, q_*\alpha)$ and $\bar{\mathcal{M}}_j(\beta) = (p_*\alpha, q_*\beta)$ for $\alpha, \beta \in H_j(X)$.

Applying Lemma 5.3, we have an isomorphism

$$\mathrm{Ker}_j\ (\lambda_2(p)_*) \cong \mathrm{Ker}_j\ (p_* + id\ )/(\mathrm{Ker}_j\ (p_* + id\ ) \cap \mathrm{Im}\ \bar{\mathcal{M}}'_j) \tag{5.7}$$

and an exact sequence

$$0 \longrightarrow \mathrm{Coker}_j\ (p_* + id\ ) \longrightarrow \mathrm{Coker}_j\ (\lambda_2(q)_*) \longrightarrow \mathrm{Coker}_j\ (0 + id\ ) \longrightarrow 0.$$

We see that $(\alpha, q_*\alpha) \in \mathrm{Ker}_j\ (p_* + id\ ) \cap \mathrm{Im}\ \bar{\mathcal{M}}'_j$ if and only if $p_*\alpha = 0, q_*\alpha = 0$. From the assumption $\mathrm{Ker}\ (q_*) \subset \mathrm{Ker}\ (p_*)$, it follows that $\mathrm{Ker}_j\ (p_* + id\ ) \cap \mathrm{Im}\ \bar{\mathcal{M}}'_j \cong \mathrm{Ker}_j\ (q_*)$.

On the other hand, $\mathrm{Coker}_j\ (0 + id\ ) \cong \mathrm{Ker}_{j-1}\ (q_*)$.

Now (ii) follows from (5.7). □

Theorem 5.8. *We have the following two exact sequences for $s : X \to A$:*

$$0 \to \mathrm{Ker}_{j-1}\ (s_*) \to H_j(J(s)) \stackrel{(\gamma_2)_*}{\to} H_j(S^1 \times A) \to \mathrm{Coker}_{j-1}\ (s_*) \to 0$$

and

$$0 \to \mathrm{Ker}_j\ (s_*) \to H_j(S^1 \times X) \stackrel{\lambda_2(s)_*}{\to} H_j(J'(s)) \to \mathrm{Coker}_j\ (s_*) \to 0.$$

Proof. To prove the first exact sequence, we apply the second exact sequence of Theorem 5.6 (ii) for $t = id_X$. Then we can prove the assertion easily.

The second one is proved symmetrically by using Theorem 5.6°. □

6. Homology of $M_{n,k}$

As in §5, we abbreviate the homology with K-coefficients as $H_*(X)$, where K is a field. We recall that $M_{n,n-1} = \{\text{1-point}\}$ and $M_{n,k} = \emptyset$ for $k \geq n$.

We think of the following assertions.

A_n: $(f^+_{n,k})_* : H_*(M_{n,k}) \to H_*(M_{n,k+1})$ is surjective if $n - k$ is even and $k \geq 1$.

B_n: $(f^-_{n,k})_* : H_*(M_{n,k}) \to H_*(M_{n,k-1})$ is injective if $n - k$ is even and $k \geq 2$.

C_n: $\mathrm{Im}\ ((f^-_{n,2})_*) \subset \mathrm{Im}\ ((f^+_{n,0})_*)$ if n is even.

C°_n: $\mathrm{Ker}\ ((f^-_{n,1})_*) \subset \mathrm{Ker}\ ((f^+_{n,1})_*)$ if n is odd.

These are the crucial parts to determine the homology of $M_{n,k}$. In the following, we prove A_n, B_n, C_n and C°_n for $n \geq 2$ by induction on n.

For the first step of the induction, we can easily check the assertions for $n = 2$ (see (6.6)). Hence in order to complete the induction, we need to prove the following:

Theorem 6.1. *The assertions $A_{n-1}, B_{n-1}, C_{n-1}$ and C°_{n-1} imply A_n, B_n, C_n and C°_n.*

Proof. (i) First we prove A_n for $k \geq 2$. By Theorem 4.7 (A)-(i), we have $f^+_{n,k} = \lambda_2(f^+_{n-1,k+1})\gamma_1(f^-_{n-1,k+1})$ for $k \geq 1$, where

$$\gamma_1(f^-_{n-1,k+1}) : M_{n,k} = I(f^-_{n-1,k+1}, f^+_{n-1,k-1}) \to J(f^-_{n-1,k+1})$$

and

$$\lambda_2(f^+_{n-1,k+1}) : J(f^-_{n-1,k+1}) \to II(f^+_{n-1,k+1}, f^-_{n-1,k+1}) = M_{n,k+1}.$$

By A_{n-1}, $(f^+_{n-1,k-1})_*$ is surjective for $k \geq 2$. Hence we can apply Theorem 5.6 (ii) with $s = f^-_{n-1,k+1}$ and $t = f^+_{n-1,k-1}$. The first exact sequence implies that $\gamma_1(f^-_{n-1,k+1})_*$ is surjective.

By B_{n-1}, $(f^-_{n-1,k+1})_*$ is injective. Hence we can apply Theorem 5.6° (ii) with $p = f^+_{n-1,k+1}$ and $q = f^-_{n-1,k+1}$. By the second exact sequence with A_{n-1} and B_{n-1}, we see that $\lambda_2(f^+_{n-1,k+1})_*$ is surjective for $k \geq 1$.

Now we see that $(f^+_{n,k})_*$ is surjective for $k \geq 2$, and hence we have shown A_n for $k \geq 2$.

(ii) Next we show A_n for $k = 1$ and C°_n. By C_{n-1}, we can apply Theorem 5.6 (ii) with $s = f^-_{n-1,2}$ and $t = f^+_{n-1,0}$. The first exact sequence is

$$(6.2) \qquad 0 \longrightarrow \mathrm{Coker}_j\ ((f^+_{n-1,0})_*) \oplus \mathrm{Ker}_{j-1}((f^+_{n-1,0})_*) \longrightarrow H_j(M_{n,1}) \xrightarrow{\gamma_1(f^-_{n-1,2})_*} H_j(J(f^-_{n-1,2})) \longrightarrow 0.$$

Since $\lambda_2(f^+_{n-1,2})_* : H_*(J(f^-_{n-1,k+1})) \to H_*(M_{n,2})$ is surjective by the argument of (i), $(f^+_{n,1})_* = \lambda_2(f^+_{n-1,2})_*\gamma_1(f^-_{n-1,2})_*$ is surjective. Thus we have shown A_n for $k = 1$.

By Theorem 4.7 (B)-(iv), we have

$$f^-_{n,1} = \gamma_2\gamma_1(f^-_{n-1,2}).$$

Since $(\gamma_2)_* : H_*(J(f^-_{n-1,2})) \to H_*(M_{n,0})$ is injective by B_{n-1}, we have

$$\mathrm{Ker}_j\ ((f^-_{n,1})_*) \cong \mathrm{Ker}_j\ (\gamma_1(f^-_{n-1,2})_*).$$

Now since $(f^+_{n,1})_* = \lambda_2(f^+_{n-1,2})_*\gamma_1(f^-_{n-1,2})_*$, we see that

$$\mathrm{Ker}\ ((f^-_{n,1})_*) \subset \mathrm{Ker}\ ((f^+_{n,1})_*).$$

Thus we have shown C°_n.

(iii) We can prove B_n and C_n similarly.

This completes the proof of Theorem 6.1. □

Now we have proved the following:

Theorem 6.3. *The assertions A_n, B_n, C_n and C°_n hold for $n \geq 2$.*

Moreover, concerning C_n and C°_n, we can add the following formulae. Since they are proved in the same way as in Theorem 6.1, we omit the proof.

Theorem 6.4. (i) *For an even n, we have*

$$\mathrm{Ker}_j\ ((f^+_{n,0})_*) \cong \mathrm{Ker}_j\ ((f^+_{n-1,1})_*).$$

(ii) *For an odd n, we have*

$$\mathrm{Coker}_j\ ((f^-_{n,1})_*) \cong \mathrm{Coker}_{j-1}\ ((f^-_{n-1,2})_*).$$

Theorem 6.3 tells us that we can apply Theorems 5.6 (i) and 5.6° (i) to $H_*(M_{n,k})$. Then by using Theorems 6.3 and 6.4, we can write down $H_*(M_{n,k})$ as follows.

Theorem 6.5. (i) *For $n - k \equiv 0$ mod 2 and $k \geq 2$,*

$$H_j(M_{n,k}) \cong H_j(M_{n-1,k}) \oplus H_{j-1}(M_{n-1,k+1}) \oplus \mathrm{Ker}_{j-1}\ ((f^+_{n-1,k-1})_*),$$

where $(f^+_{n-1,k-1})_* : H_*(M_{n-1,k-1}) \to H_*(M_{n-1,k})$ is surjective.

(ii) *For $n - k \equiv 1$ mod 2 and $k \geq 2$,*

$$H_j(M_{n,k}) \cong H_j(M_{n-1,k+1}) \oplus \mathrm{Coker}_j\ ((f^-_{n-1,k})_*) \oplus H_{j-1}(M_{n-1,k}),$$

where $(f^-_{n-1,k})_* : H_*(M_{n-1,k}) \to H_*(M_{n-1,k-1})$ is injective.

(iii)

$$\begin{aligned} H_j(M_{2m+1,1}) \cong{}& H_j(M_{2m,1}) \oplus \mathrm{Coker}_j\ ((f^+_{2m,0})_*) \\ & \oplus H_{j-1}(M_{2m,2}) \oplus \mathrm{Ker}_{j-1}\ ((f^+_{2m,0})_*), \end{aligned}$$

where

$$\mathrm{Ker}_{j-1}\ ((f^+_{2m,0})_*) \cong \mathrm{Ker}_{j-1}\ ((f^+_{2m-1,1})_*)$$

and $(f^+_{2m-1,1})_* : H_*(M_{2m-1,1}) \to H_*(M_{2m-1,2})$ *is surjective.*

(iv)

$$\begin{aligned} H_j(M_{2m,1}) \cong{}& H_j(M_{2m-1,2}) \oplus \mathrm{Coker}_j\ ((f^-_{2m-1,1})_*) \oplus H_{j-1}(M_{2m-1,1}) \\ & \oplus \mathrm{Ker}_{j-1}\ ((f^-_{2m-1,1})_*), \end{aligned}$$

where

$$\mathrm{Coker}_j\ ((f^-_{2m-1,1})_*) \cong \mathrm{Coker}_{j-1}\ ((f^-_{2m-2,2})_*)$$

and $(f^-_{2m-2,2})_* : H_*(M_{2m-2,2}) \to H_*(M_{2m-2,1})$ *is injective.*

Now we give recurrence relations for $H_*(M_{n,k}; K)$. When we have fixed a field K, we denote the Poincaré polynomial of $M_{n,k}$ with K-coefficients by $PS(n,k)$. Thus $PS(n,k) = \sum_\lambda \dim_K H_\lambda(M_{n,k}; K)t^\lambda$. We note the following initial conditions.

(6.6) (i) $PS(1,k) = 0$ for $k \in \mathbf{N} \cup \{0\}$.

(ii) $PS(2,1) = 1$, while $PS(2,k) = 0$ for $k \in \mathbf{N} \cup \{0\}$ and $k \neq 1$.

Then our recurrence relations are given by the following:

Theorem 6.7. (I) *For* $m \geq 1$, *we have*

$$\begin{aligned} PS(2m+1,0) ={}& (1+t)PS(2m,1), \\ PS(2m+1,1) ={}& -(1+t)PS(2m-1,2) + 2PS(2m,1) + tPS(2m,2), \\ PS(2m+1,2i+1) ={}& tPS(2m,2i) + (1-t)PS(2m,2i+1) + tPS(2m,2i+2), \\ PS(2m+1,2i) ={}& PS(2m,2i-1) + (t-1)PS(2m,2i) + PS(2m,2i+1), \\ PS(2m+1,2m) ={}& 1. \end{aligned}$$

(II) *For* $m \geq 2$, *we have*

$$\begin{aligned} PS(2m,0) ={}& (1+t)PS(2m-1,1), \\ PS(2m,1) ={}& PS(2m-1,2) + tPS(2m-1,1) + (1+t)t(PS(2m-2,1) \\ & - PS(2m-2,2)) + t(PS(2m-1,1) - PS(2m-1,0)), \\ PS(2m,2i+1) ={}& PS(2m-1,2i) + (t-1)PS(2m-1,2i+1) \\ & + PS(2m-1,2i+2), \\ PS(2m,2i) ={}& tPS(2m-1,2i-1) + (1-t)PS(2m-1,2i) \\ & + tPS(2m-1,2i+1), \\ PS(2m,2m-1) ={}& 1. \end{aligned}$$

Proof. We prove only $PS(2m,1)$. From Theorem 6.5 (iv), we have

$$\begin{aligned} PS(2m,1) ={}& PS(2m-1,2) + PS(\mathrm{Coker}\ ((f^-_{2m-1,1})_*)) + tPS(2m-1,1) \\ & + tPS(\mathrm{Ker}\ ((f^-_{2m-1,1})_*)) \end{aligned}$$

and

$$PS(\text{Coker } ((f^-_{2m-1,1})_*)) = t(PS(2m-2,1) - PS(2m-2,2)).$$

Note that we have

$$\begin{aligned}\dim \text{Ker}_j\ (f^-_{2m-1,1})_*) =& \dim \text{Coker}_j\ (f^-_{2m-1,1})_*) + \dim H_j(M_{2m-1,1}) \\ &- \dim H_j(M_{2m-1,0})\end{aligned}$$

for $(f^-_{2m-1,1})_* : H_j(M_{2m-1,1}) \to H_j(M_{2m-1,0})$. Hence we have

$$\begin{aligned}PS(\text{Ker } ((f^-_{2m-1,1})_*)) =& t(PS(2m-2,1) - PS(2m-2,2)) + PS(2m-1,1) \\ &- PS(2m-1,0).\end{aligned}$$

Now the result follows easily. □

Now the solution of the recurrence relations of Theorem 6.7 under the initial conditions (6.6) is given by the following:

Theorem 6.8. (I) *For $m \geq 0$, we have*

$$\begin{aligned}PS(2m+1,0) =& \sum_{\lambda=0}^{m-2} \binom{2m}{\lambda} t^\lambda + \left\{ \binom{2m}{m-1} + \binom{2m-1}{m-2} \right\} t^{m-1} \\ &+ \left\{ \binom{2m}{m-1} + \binom{2m-1}{m-3} \right\} t^m + \sum_{\lambda=m+1}^{2m-2} \binom{2m}{\lambda+2} t^\lambda, \\ PS(2m+1,1) =& \sum_{\lambda=0}^{m-2} \binom{2m}{\lambda} t^\lambda + 2\binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-2} \binom{2m}{\lambda+2} t^\lambda, \\ PS(2m+1,2i+1) =& \sum_{\lambda=0}^{m-i-1} \binom{2m}{\lambda} t^\lambda + \sum_{\lambda=m+i-1}^{2m-2} \binom{2m}{\lambda+2} t^\lambda \ \ (1 \leq i \leq m-1), \\ PS(2m+1,2i) =& \sum_{\lambda=0}^{m-i} \binom{2m}{\lambda} t^\lambda + \sum_{\lambda=m+i-1}^{2m-2} \binom{2m}{\lambda+2} t^\lambda \ \ (1 \leq i \leq m).\end{aligned}$$

(II) *For $m \geq 1$, we have*

$$\begin{aligned}PS(2m,0) =& \sum_{\lambda=0}^{m-3} \binom{2m-1}{\lambda} t^\lambda + \left\{ 2\binom{2m-2}{m-2} + \binom{2m-3}{m-4} \right. \\ &+ \left. \binom{2m-3}{m-3} \right\} (t^{m-2} + t^{m-1}) + \sum_{\lambda=m}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda, \\ PS(2m,1) =& \sum_{\lambda=0}^{m-2} \binom{2m-1}{\lambda} t^\lambda + \binom{2m}{m-1} t^{m-1} + \sum_{\lambda=m}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda, \\ PS(2m,2i+1) =& \sum_{\lambda=0}^{m-i-1} \binom{2m-1}{\lambda} t^\lambda + \sum_{\lambda=m+i-1}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda \ \ (1 \leq i \leq m-1), \\ PS(2m,2i) =& \sum_{\lambda=0}^{m-i-1} \binom{2m-1}{\lambda} t^\lambda + \sum_{\lambda=m+i-2}^{2m-3} \binom{2m-1}{\lambda+2} t^\lambda \ \ (1 \leq i \leq m-1).\end{aligned}$$

Proof. We can prove Theorem 6.8 easily by induction on n (= the number of vertices, i.e., $n = 2m$ or $2m + 1$). □

Theorem 6.8 shows that $PS(n, k)$ does not depend on the coefficient field K. Hence we see that $H_*(M_{n,k}; \mathbf{Z})$ is a torsion free module, and Theorem B follows.

Again by Theorem 6.8, we have determined $H_*(M_{n,k}; \mathbf{Z})$. In particular, we have Theorem C.

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References

[1] A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, 1972. MR **51:**1825

[2] J.-C. Hausmann, *Sur la topologie des bras articulés*, Algebraic Topology, Poznań 1989, Lecture Notes in Math., vol. 1474, Springer-Verlag, 1991, pp. 146-159. MR **93a:**57035

[3] T. Havel, *Some examples of the use of distances as coordinates in Euclidean geometry*, Journal of Symbolic Computation **11** (1991), 579-593. MR **92j:**51033

[4] Y. Kamiyama, *An elementary proof of a theorem of T. F. Havel*, Ryukyu Math. J. **5** (1992), 7-12. MR **94a:**52044

[5] Y. Kamiyama, *Topology of equilateral polygon linkages*, Top. and its Applications **68** (1996), 13-31. MR **96j:**52041

[6] M. Kapovich and J. Millson, *On the moduli space of polygons in the Euclidean plane*, Journal of Diff. Geometry **42** (1995), 133-164. MR **98b:**52019

[7] M. Kato, *Topology of k-regular spaces and algebraic sets*, Manifolds-Tokyo 1973, Univ. of Tokyo Press, Tokyo, 1975, pp. 153-159. MR **54:**6149

[8] J. Milnor, *Morse theory*, Ann. of Math. Studies, vol. 51, Princeton Univ. Press, Princeton, 1963. MR **29:**634

[9] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, vol. 61, Princeton Univ. Press, Princeton, 1968. MR **39:**969

[10] I. Schoenberg, *Linkages and distance geometry.* I. *Linkages*, Indag. Math. **31** (1969), 42-52. MR **39:**7512

[11] T. Toma, *An analogue of a theorem of T. F. Havel*, Ryukyu Math. J. **6** (1993), 69-77; *Corrections*, ibid. **8** (1995), 95-96. MR **95g:**52038

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